Synopsis of material from EGA 0 (IV)
§14: Combinatorial dimension of a topological space
§16: Dimension and depth for Noetherian local rings, 16.1–16.3

14.1. **Combinatorial dimension of a topological space.**

(14.1.1). A chain $i_0 < i_1 < \cdots < i_n$ in an ordered set has *length* n.

Definition (14.1.2). — The *(combinatorial) dimension* $\dim(X)$ of a topological space X is the supremum of the lengths of chains of irreducible closed subsets of X. The *dimension* $\dim_x(X)$ *of* X *at* x is $\inf_U \dim(U)$, over open sets $x \in U \subseteq X$.

Clearly $\dim(X) = \sup_\alpha \dim(X_\alpha)$, where X_α ranges over the irreducible components of X.

Definition (14.1.3). — X is *equidimensional* if all irreducible components have the same dimension.

Proposition (14.1.4). — *(i) For every closed* $Y \subseteq X$*, we have* $\dim(Y) \leq \dim(X)$*. (ii) If* X *is a finite union of closed subsets* X_i*, then* $\dim(X) = \sup_i \dim(X_i)$.

Corollary (14.1.5). — *Let* U *be a neighborhood of* $x \in X$ *and suppose that* U *is the union of finitely many closed subsets* $Y_i \subseteq U$*, each containing* x*. Then*

(14.1.5.1) $$\dim_x(X) = \sup_i \dim_x(Y_i).$$

Proposition (14.1.6). — *For any* X *we have* $\dim(X) = \sup_{x\in X} \dim_x(X)$.

Corollary (14.1.7). — *If* (X_α) *is either an open covering of* X*, or a locally finite closed covering of* X*, then* $\dim(X) = \sup_\alpha \dim(X_\alpha)$.

Corollary (14.1.8). — *If* X *is a Noetherian* T_0 *space and* $F \subseteq X$ *the set of closed points, then* $\dim(X) = \sup_{x\in F} \dim_x(X)$.

Proposition (14.1.9). — *A Noetherian* T_0 *space* X *has* $\dim(X) = 0$ *iff* X *is finite and discrete.*

Corollary (14.1.10). — *A point* $x \in X$ *in a Noetherian* T_0 *space is isolated iff* $\dim_x(X) = 0$.

Proposition (14.1.11). — *The function* $x \mapsto \dim_x(X)$ *is upper-semicontinuous* [a function f is *upper-semicontinuous* if $\{x : f(x) \leq m\}$ is open for all m].

Remark (14.1.12). — It is possible to have $\dim(f(X)) > \dim(X)$ for a continuous map f.

14.2. **Codimension of a closed subset.**

Definition (14.2.1). — The *(combinatorial) codimension* $\operatorname{codim}(Y, X)$ of an irreducible closed $Y \subseteq X$ is the supremum of the lengths of chains of irreducibles whose minimum element is Y. For any closed Y, define $\operatorname{codim}(Y, X)$ to be the infimum of the codimensions of the irreducible components of Y. If all minimal irreducible closed subsets $Y \subseteq X$ have the same codimension, X is called *equicodimensional*.

If (X_α) are the irreducible components of X, and Y_β those of Y, then every Y_β is contained in some X_α, and

$$\operatorname{codim}(Y, X) = \inf_\beta(\sup_\alpha(\operatorname{codim}(Y_\beta, X_\alpha)), \tag{14.2.1.1}$$

the indices ranging over α, β such that $Y_\beta \subseteq X_\alpha$.

Proposition (14.2.2). — *(i) If Φ is the set of all irreducible closed subsets of X, then*

$$\dim(X) = \sup_{Y \in \Phi}(\operatorname{codim}(Y, X)). \tag{14.2.2.1}$$

(ii) For every non-empty closed $Y \subseteq X$,

$$\dim(Y) + \operatorname{codim}(Y, X) \leq \dim(X). \tag{14.2.2.2}$$

(iii) For every three closed subsets $Y \subseteq Z \subseteq T$,

$$\operatorname{codim}(Y, Z) + \operatorname{codim}(Z, T) \leq \operatorname{codim}(Y, T). \tag{14.2.2.3}$$

(iv) A closed subset $Y \subseteq X$ has $\operatorname{codim}(Y, X) = 0$ *iff Y contains an irreducible component of X.*

Proposition (14.2.3). — *Let $Y \subseteq X$ be closed, $U \subseteq X$ open. Then*

$$\operatorname{codim}(Y \cap U, U) \geq \operatorname{codim}(Y, X), \tag{14.2.3.1}$$

with equality iff $\operatorname{codim}(Y, X) = \inf_\alpha(\operatorname{codim}(Y_\alpha, X))$ *where Y_α ranges over the irreducible components of Y which meet U.*

Definition (14.2.4). — Let $Y \subseteq X$ be closed, $x \in X$ a point. Define $\operatorname{codim}_x(Y, X) = \sup_U(\operatorname{codim}(Y \cap U, U))$, where U ranges over neighborhoods of x.

Note that if $x \notin Y$, then $\operatorname{codim}_x(Y, X) = +\infty$.

Proposition (14.2.5). — *If Y is the union of finitely many closed subsets $Y_i \subseteq X$, then*

$$\operatorname{codim}(Y, X) = \inf_i \operatorname{codim}(Y_i, X). \tag{14.2.5.1}$$

Corollary (14.2.6). — *Let Y be a locally Noetherian subspace of X.*

(i) For every $x \in X$, only finitely many irreducible components Y_i of Y contain x, and $\operatorname{codim}_x(Y, X) = \inf_i \operatorname{codim}(Y_i, X)$.

(ii) The function $x \mapsto \operatorname{codim}_x(Y, X)$ is lower semi-continuous.

14.3. **The chain condition.**

(14.3.1). A chain $Z_0 \subset \cdots \subset Z_n$ of irreducible closed subsets is *saturated* if no additional element $Z_k \subset Z' \subset Z_{k+1}$ can be inserted.

Proposition (14.3.2). — *Assume that* $\operatorname{codim}(Y,Z) < \infty$ *for all irreducibles* $Y \subseteq Z$ *in* X. *The following conditions are equivalent.*

(a) Any two saturated chains with the same endpoints have the same length.

(b) For all irreducibles $Y \subseteq Z \subseteq T$, *there holds*

(14.3.2.1) $$\operatorname{codim}(Y,Z) + \operatorname{codim}(Z,T) = \operatorname{codim}(Y,T).$$

A space X satisfying these *chain conditions* is called *catenary.*

Proposition (14.3.3). — *Let* X *be a finite-dimensional Noetherian* T_0 *space. The following are equivalent.*

(a) Any two maximal chains of irreducibles have the same length.

(b) X *is equidimensional, equicodimensional, and catenary.*

(c) X *is equidimensional and for all irreducibles* $Y \subseteq Z \subseteq X$,

(14.3.3.1) $$\dim(Z) = \dim(Y) + \operatorname{codim}(Y,Z).$$

(d) X *is equicodimensional and for all irreducibles* $Y \subseteq Z \subseteq X$,

(14.3.3.2) $$\operatorname{codim}(Y,X) = \operatorname{codim}(Y,Z) + \operatorname{codim}(Z,X).$$

A space satisfying the above conditions is *biequidimensional*

Corollary (14.3.4). — *Let* X *be a biequidimensional Noetherian* T_0 *space. Then for every closed point* $x \in X$ *and irreducible component* Z *of* X,

(14.3.4.1) $$\dim(X) = \dim(Z) = \operatorname{codim}(\{x\}, X) = \dim_x(X).$$

Corollary (14.3.5). — *Let* X *be a Noetherian* T_0 *space. If* X *is biequidimensional, then so is any union of irreducible components of* X, *and any irreducible closed subset of* X. *Moreover, every closed* $Y \subseteq X$ *satisfies*

(14.3.5.1) $$\dim(Y) + \operatorname{codim}(Y,X) = \dim(X).$$

Remark (14.3.6). — Various of the results above on chain conditions apply to any partially ordered set whatsoever, not just the set of irreducible closed subsets of a topological space.

16. Dimension and depth in local Noetherian rings

16.1. **Dimension of a ring.**

(16.1.1). The *(Krull) dimension* $\dim(A)$ of a ring A is $\dim(\operatorname{Spec}(A))$; that is, the supremum of the lengths of chains of prime ideals in A. In particular, $\dim(A) \geq 0$ if $A \neq 0$. A non-zero Artinian ring (*e.g.* a field) has dimension zero. A Dedekind ring [principal ideal domain] which is not a field, such as $\mathbb{Z}$ or $k[t]$ (k a field) has dimension 1. A Noetherian ring need not have finite dimension.

If $(\mathfrak{p}_\alpha)$ is the set of minimal primes of A, then

(16.1.1.1) $$\dim(A) = \sup_\alpha \dim(A/\mathfrak{p}_\alpha).$$

(16.1.2). For any ideal $\mathfrak{a} \subseteq A$, we have

$$\dim(A/\mathfrak{a}) \leq \dim(A), \tag{16.1.2.1}$$

and if $\mathfrak{a}$ is not contained in any minimal prime, then

$$\dim(A/\mathfrak{a}) < \dim(A). \tag{16.1.2.2}$$

(16.1.3). For any multiplicative set $S \subseteq A$, we have

$$\dim(S^{-1}A) \leq \dim(A). \tag{16.1.3.1}$$

For any prime ideal $\mathfrak{p} \subseteq A$, we have

$$\dim(A_{\mathfrak{p}}) = \operatorname{codim}(V(\mathfrak{p}), \operatorname{Spec}(A)), \tag{16.1.3.2}$$

also denoted $\operatorname{ht}(\mathfrak{p})$ and called the *height* of $\mathfrak{p}$. More generally, for any ideal $\mathfrak{a}$, define

$$\operatorname{ht}(\mathfrak{a}) = \operatorname{codim}(V(\mathfrak{a}), \operatorname{Spec}(A)) \tag{16.1.3.3}$$

If $(\mathfrak{p}_\lambda)$ is the set of primes minimally containing $\mathfrak{a}$, then $\operatorname{ht}(\mathfrak{a}) = \inf_\lambda \operatorname{ht}(\mathfrak{p}_\lambda)$. Clearly, $\dim(A) = \sup_{\mathfrak{m}} \dim(A_{\mathfrak{m}})$, where $\mathfrak{m}$ ranges over the maximal ideals of A.

(16.1.4). By (14.2.2.2), we have for any prime ideal $\mathfrak{p}$,

$$\dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) \leq \dim(A). \tag{16.1.4.1}$$

We call A *catenary*, *equidimensional*, *equicodimensional*, *biequidimensional* if $\operatorname{Spec}(A)$ is so. If A is Noetherian and biequidimensional, equality holds in (16.1.4.1). If A is catenary, so are $A/\mathfrak{a}$ and $S^{-1}A$. Moreover A is catenary if and only if every $A_{\mathfrak{p}}$ is; *i.e.*, iff for all primes $\mathfrak{q} \subseteq \mathfrak{p}$,

$$\dim(A_{\mathfrak{q}}) + \dim(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = \dim(A_{\mathfrak{p}}). \tag{16.1.4.2}$$

Proposition (16.1.5). — *If B is an A-algebra integral over A, then* $\dim(B) \leq \dim(A)$, *with equality if A is a subring of B.*

[This follows from the "going-up" theorem—see also Liu 2.5.10.]

Proposition (16.1.6). — *Let B be an integral domain, A a local subring of B with maximal ideal $\mathfrak{m}$. If A is integrally closed and B is integral over A, then* $\dim(B_{\mathfrak{n}}) = \dim(A)$ *for every maximal ideal $\mathfrak{n}$ of B.*

[This follows from the "going-down" theorem.]

(16.1.7). If M is an A-module, we define $\dim(M) = \dim(A/\operatorname{ann}(M))$. If M is finitely generated, then $\operatorname{Supp}(M)$ is closed in $\operatorname{Spec}(A)$, and $\dim(M) = \dim(\operatorname{Supp}(M))$. If $N \subseteq M$, then $\dim(N) \leq \dim(M)$ and $\dim(M/N) \leq \dim(M)$. If S is a multiplicative set, then $\dim_{S^{-1}A}(S^{-1}M) \leq \dim_A(M)$. We have $\dim_A(M) = \sup_{\mathfrak{m}} \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$ where $\mathfrak{m}$ ranges over the maximal ideals of A. We call M *catenary*, *equidimensional*, etc., if the same hold for $A/\operatorname{ann}(M)$.

Proposition (16.1.8). — *Let M be a finitely generated A-module, $\mathfrak{p} \subseteq A$ a prime ideal. Then*

$$\dim_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim_A(M/\mathfrak{p}M) \leq \dim_A(M). \tag{16.1.8.1}$$

Note that $\dim_A(M/\mathfrak{p}M) = \dim_{A/\mathfrak{p}}(M/\mathfrak{p}M)$.

Proposition (16.1.9). — *Let A be Noetherian, B an A-algebra, M a B-module such that $M_{[A]}$ is finitely generated. Then*

$$\dim_A(M_{[A]}) = \dim_B(M). \tag{16.1.9.1}$$

(16.1.10). If A is Noetherian and $M \neq 0$ is an A-module of finite length, then $\mathrm{Supp}(M)$ is discrete, hence $\dim(M) = 0$; and conversely.

16.2. **Dimension of a semi-local Noetherian ring.** [See Liu §2.5.2 for another approach, taking Krull's theorem (16.3.2) as the starting point.]

(16.2.1). Let A be Noetherian and semi-local [A has finitely many maximal ideals], with [Jacobson] radical $\mathfrak{r}$. An *ideal of definition* is an ideal $\mathfrak{q}$ such that some power $\mathfrak{r}^n \subseteq \mathfrak{q}$; or equivalently, all primes containing are maximal; or $A/\mathfrak{q}$ is Artinian. Then $M/\mathfrak{q}^n M$ has finite length for any finitely generated A-module M. There is a polynomial $P_\mathfrak{q}(M, n)$ in n with rational coefficients, the *Hilbert-Samuel polynomial*, such that $l(M/\mathfrak{q}^n M) = P_\mathfrak{q}(M, n)$ for $n >> 0$. Denote by $d(M)$ the degree of $P_\mathfrak{q}(M, n)$.

Lemma (16.2.2.1). — *If (M_n) is a $\mathfrak{q}$-stable filtration, then for $n >> 0$, $l(M/M_n)$ is given by a polynomial in n with the same degree $d(M)$ and leading coefficient as $P_\mathfrak{q}(M, n)$.*

Lemma (16.2.2.2). — *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $P_\mathfrak{q}(M, n) - (P_\mathfrak{q}(M', n) + P_\mathfrak{q}(M'', n))$ has degree $< d(M') \leq d(M)$.*

Theorem (16.2.3). — *Assume $M \neq 0$, and let $s(M)$ be the smallest n such that there exist $x_1, \ldots, x_n \in \mathfrak{r}$ for which $M/(x_1 M + \cdots + x_n M)$ has finite length. Then*

$$d(M) = s(M) = \dim(M). \tag{16.2.3.1}$$

Corollary (16.2.4). — *Let $\widehat{A} = \varprojlim_n A/\mathfrak{r}^n$ be the $\mathfrak{r}$-adic completion of A. Then $\dim_{\widehat{A}}(\widehat{M}) = \dim_A(M)$.*

Corollary (16.2.5). — *The dimension of a semi-local Noetherian ring A is finite and equal to the smallest number of elements of $\mathfrak{r}$ which generate an ideal of definition.*

Corollary (16.2.6). — *Let $(A, \mathfrak{m})$ be a Noetherian local ring, with $k = A/\mathfrak{m}$. Then $\dim(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.*

16.3. **Systems of parameters in a Noetherian local ring.**

Proposition (16.3.1). — *Let $\mathfrak{p}$ be a prime ideal in a Noetherian ring A. The following are equivalent:*

(a) $\mathrm{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}) \leq n$.

(b) There exist n elements $x_i \in \mathfrak{p}$ such that $\mathfrak{p}$ is minimal over $\mathfrak{a} = (x_1, \ldots, x_n)$, i.e., $V(\mathfrak{p})$ is an irreducible component of $V(\mathfrak{a})$.

The special case $n = 1$ is *Krull's principal ideal theorem*:

Corollary (16.3.2). — *A prime ideal in a Noetherian ring has height ≤ 1 iff it is minimal over a principal ideal.*

Corollary (16.3.3). — *Let A be a Noetherian domain. The following are equivalent:*

(a) A is semi-local of dimension ≤ 1.

(b) There exists $f \neq 0$ in A such that A_f is a field.

Proposition (16.3.4). — *Let A be a semi-local Noetherian ring with radical $\mathfrak{r}$, M a finitely generated A-module, $\mathfrak{p}_i$ the elements of* $\mathrm{Supp}(M)$ *such that* $\dim(A/\mathfrak{p}_i) = \dim(M)$. *Then the $\mathfrak{p}_i$ are minimal in* $\mathrm{Ass}(M)$, $\dim(M/\mathfrak{p}_i M) = \dim(M)$, *and for every $x \in \mathfrak{r}$,*

(16.3.4.1) $$\dim(M/xM) \geq \dim(M) - 1,$$

with equality iff $x \notin \bigcup_i \mathfrak{p}_i$.

Corollary (16.3.5). — *With the notation of (16.3.4), for any ideal $\mathfrak{a} \subseteq \mathfrak{p}_i$, we have* $\dim(M/\mathfrak{a}M) = \dim(M)$, *and if* $\dim(M/xM) = \dim(M) - 1$, *then* $\dim(N/xN) = \dim(N) - 1$, *where $N = M/\mathfrak{a}M$.*

Definition (16.3.6). — Let A be a semi-local Noetherian ring with radical $\mathfrak{r}$, M a finitely generated A-module, $n = \dim(M)$. A sequence of n elements $x_1, \ldots, x_n$ such that $M/(x_1 M + \cdots + x_n M)$ has finite length is a *system of parameters* for M.

A system of parameters for M is a system of parameters for $A/\mathrm{ann}(M)$ and conversely. A system of parameters always exists, by (16.2.3).

Proposition (16.3.7). — *Let A be a semi-local Noetherian ring with radical $\mathfrak{r}$, M a finitely generated A-module, $x_1, \ldots, x_k \in \mathfrak{r}$. Then*

(16.3.7.1) $$\dim(M/(x_1 M + \cdots + x_k M)) \geq \dim(M) - k,$$

with equality iff (x_i) is part of a system of parameters for M.

Proposition (16.3.8). — *Let* $X = \mathrm{Spec}(A)$, *where A is a semi-local Noetherian ring; $\mathfrak{a} \subseteq A$, $\mathfrak{a} \neq A$ an ideal such that* $\mathrm{codim}(V(\mathfrak{a}), X) = r > 0$. *Then there exist $x_1, \ldots, x_r$ forming part of a system of parameters for A, such that* $\mathrm{codim}(V(\mathfrak{a}), V(x_1, \ldots, x_r)) = 0$.

Proposition (16.3.9). — *Let $\phi\colon A \to B$ be a local homomorphism of local rings, $k = A/\mathfrak{m}$ the residue field.*

(i) We have

(16.3.9.1) $$\dim(B) \leq \dim(A) + \dim(B \otimes_A k).$$

(ii) Let M, N be finitely generated, non-zero A, B-modules. Then

$$\dim_B(M \otimes_A N) \leq \dim_A(M) + \dim_{B\otimes_A k}(N \otimes_A k). \tag{16.3.9.2}$$

Corollary (16.3.10). — *In (16.3.9), if $\mathfrak{m}B$ is an ideal of definition of B, then* $\dim(B) \leq \dim(A)$. *If A is a domain and equality holds, then ϕ is injective.*