

SYNOPSIS OF MATERIAL FROM EGA CH. 0
§I: RINGS OF FRACTIONS

1.0 Rings and algebras.

(1.0.1) All rings have 1, and are commutative unless otherwise specified. By default, modules over non-commutative rings are left modules.

(1.0.2) Given a homomorphism of (possibly non-commutative) rings $\phi: A \rightarrow B$, every B -module M has naturally an A -module structure. Given another A -module L and A -module morphism $f: L \rightarrow M$, the pair (ϕ, f) is called a *di-homomorphism* from (A, L) to (B, M) . This makes pairs (ring, module) the objects of a category.

(1.0.3) Given a (left) ideal $I \subseteq A$, one writes BI for the (left) ideal $B\phi(I) \subseteq B$, which is also the image of the canonical homomorphism $B \otimes_A I \rightarrow B$. Similarly for right ideals.

(1.0.4) Let A be commutative. An A -algebra is a ring B with a homomorphism $\phi: A \rightarrow B$ whose image is in the center of B . Then for every $I \subseteq A$, the ideal $IB = BI$ is a two-sided ideal of B , and if M is a B -module, $IM = BIM$ is a submodule.

(1.0.5) An A -algebra B is *integral* over A if each element $b \in B$ is the root of a monic polynomial over A ; equivalently, b is contained in a subalgebra of B which is a finitely-generated A -module. If B is commutative, this is equivalent to every finitely-generated subalgebra of B being a finitely-generated A -module. So in this case, B is integral and of finite-type over A iff B is a f.-g. A -module.

(1.0.6) A/\mathfrak{p} is an integral domain iff \mathfrak{p} is a prime ideal. Note: the unit ideal (1) is by definition *not* a prime ideal. Correspondingly, the zero ring $A = \{0\}$, with $1 = 0$, is not an integral domain, and its set of prime ideals is *empty*.

(1.0.7) A is a *local ring* if A has a unique maximal proper ideal \mathfrak{m} . Equivalently, every $x \notin \mathfrak{m}$ is a unit in A . A homomorphism $\phi: A \rightarrow B$ of local rings $(A, \mathfrak{m}), (B, \mathfrak{n})$ is *local* if $\phi(\mathfrak{m}) \subseteq \mathfrak{n}$, which is equivalent to $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$. The composition of local homomorphisms is local.

1.1 Radical of an ideal. Nilradical and radical of a ring.

(1.1.1) The *radical* of an ideal $\mathfrak{a} \subseteq A$ is the ideal $\sqrt{\mathfrak{a}} = \{x | \exists n x^n \in \mathfrak{a}\}$. One has $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$, $\sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$, and $\phi^{-1}(\sqrt{\mathfrak{a}}) \subseteq \sqrt{\phi^{-1}(\mathfrak{a})}$ for any ring homomorphism $\phi: A' \rightarrow A$.

If $\mathfrak{a} = \sqrt{\mathfrak{a}}$, we say that \mathfrak{a} is a *radical ideal*. This is true iff \mathfrak{a} is an intersection of prime ideals, and in general $\sqrt{\mathfrak{a}}$ is the intersection of the prime ideals $\mathfrak{p} \supseteq \mathfrak{a}$. If A is Noetherian there are finitely many minimal such primes.

The ideal $\sqrt{(0)}$ is called the *nilradical* of A . If $\sqrt{(0)} = (0)$, A is *reduced*. In other words, A has no non-zero nilpotent elements. A/\mathfrak{a} is reduced iff \mathfrak{a} is a radical ideal. A subring of a reduced ring is reduced.

(1.1.2) The (Jacobson) *radical* $\mathfrak{R}(A)$ of a (possibly noncommutative) ring A is the intersection of its maximal left ideals (equivalently, right ideals). The radical of $A/\mathfrak{R}(A)$ is zero.

1.2 Modules and rings of fractions.

(1.2.1) A subset $S \subseteq A$ is *multiplicative* if $1 \in S$ and S is closed under products. Examples: (1) $S_f = \{f^n | n \geq 0\}$, (2) $A \setminus \mathfrak{p}$ for a prime ideal \mathfrak{p} .

(1.2.2) [Liu p. 10] Given a multiplicative set S and an A -module M , define $S^{-1}M$ to be the quotient of $M \times S$ by the equivalence relation

$$(m_1, s_1) \equiv (m_2, s_2) \quad \text{iff there exists } s \in S \text{ such that } s(s_1m_2 - s_2m_1) = 0.$$

Write m/s for the equivalence class of (m, s) . One has a canonical map $i_M^S: M \rightarrow S^{-1}M$, $i(m) = m/1$. In general, i_M^S is neither surjective nor injective. Its kernel is the set of elements $m \in M$ such that $sm = 0$ for some $s \in S$.

$S^{-1}A$ is a ring, $S^{-1}M$ is an $S^{-1}A$ -module, the canonical map i_A^S is a ring homomorphism, and the canonical map i_M^S is an A -module homomorphism [in other words, (i_A^S, i_M^S) is a di-homomorphism $(A, M) \rightarrow (S^{-1}A, S^{-1}M)$].

(1.2.3) When $S = S_f$, we write A_f, M_f instead of $S_f^{-1}A, S_f^{-1}M$. The ring A_f is isomorphic to $A[X]/(fX - 1)$. If f is a unit, then $A_f = A, M_f = M$; if f is nilpotent, then $A_f = \{0\}, M_f = \{0\}$.

When $S = A \setminus \mathfrak{p}$, we write $A_{\mathfrak{p}}, M_{\mathfrak{p}}$. In this case, $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}} (= S^{-1}\mathfrak{p})$, and $\mathfrak{p} = (i_A^S)^{-1}(\mathfrak{m})$. Passing to the quotient, i_A^S induces a ring homomorphism from A to the residue field $A_{\mathfrak{p}}/\mathfrak{m}$, which is identified with the field of fractions of the integral domain A/\mathfrak{p} .

(1.2.4) Universal property: any ring homomorphism $\phi: A \rightarrow B$ such that $\phi(S)$ consists of units in B factors uniquely through $i_A^S: A \rightarrow S^{-1}A$. Under the same hypotheses if N is a B -module, any A -module homomorphism $M \rightarrow N$ factors uniquely through $i_M^S: M \rightarrow S^{-1}M$.

(1.2.5) [Liu 1.2.10] One has a canonical isomorphism $S^{-1}M \cong S^{-1}A \otimes_A M$ such that $m/s \leftrightarrow 1/s \otimes m$. More precisely, this gives a natural isomorphism between the functors $S^{-1}(-)$ and $S^{-1}A \otimes_A -$ from A -modules to $S^{-1}A$ -modules—see (1.3.1).

(1.2.6) For every ideal $\mathfrak{a}' \subseteq S^{-1}A$, $\mathfrak{a} = (i_A^S)^{-1}(\mathfrak{a}')$ is an ideal of A , and one has $\mathfrak{a}' = \mathfrak{a}S^{-1}A = S^{-1}\mathfrak{a}$. This gives a bijective, inclusion-preserving correspondence between the prime ideals of $S^{-1}A$ and those prime ideals $\mathfrak{p} \subseteq A$ such that $\mathfrak{p} \cap S = \emptyset$. The local rings $A_{\mathfrak{p}}$ and $(S^{-1}A)_{S^{-1}\mathfrak{p}}$ are canonically isomorphic.

(1.2.7) If A is an integral domain, then i_A^S is injective provided $0 \notin S$, and $S^{-1}A$ is a subring containing A of the fraction field K of A . In particular, $A_{\mathfrak{p}}$ is local with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$, and $\mathfrak{p}A_{\mathfrak{p}} \cap A = \mathfrak{p}$.

(1.2.8) If A is reduced, then so is $S^{-1}A$.

1.3 Functorial properties.

(1.3.1-2) [Liu 1.2.11] $M \mapsto S^{-1}M$ is an exact functor from A -modules to $S^{-1}A$ -modules. In particular, if $N, P \subseteq M$ are submodules, then so are $S^{-1}N, S^{-1}P \subseteq S^{-1}M$, and $S^{-1}(-)$ commutes with \cap and $+$.

(1.3.3) $S^{-1}(-)$ commutes with direct limits.

(1.3.4) There is a natural isomorphism of functors

$$(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \cong S^{-1}(M \otimes_A N)$$

such that $(m/s) \otimes (n/t) \leftrightarrow (m \otimes n)/st$.

(1.3.5) There is a natural transformation between functors

$$S^{-1} \text{Hom}(M, N) \rightarrow \text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

sending ψ/s to the homomorphism $m/t \mapsto \psi(m)/st$. If M is finitely presented (in particular, if A is Noetherian and M is finitely generated), this is an isomorphism.

1.4 Change of multiplicative sets.

(1.4.1-2) Given multiplicative sets $S \subseteq T \subseteq A$, there is a canonical homomorphism $\rho^{S,T}: S^{-1}A \rightarrow T^{-1}A$, and for any A -module M , a canonical homomorphism of $S^{-1}A$ -modules $S^{-1}M \rightarrow T^{-1}M$ (under the identification $S^{-1}M = S^{-1}A \otimes_A M$, the latter is $\rho^{S,T} \otimes 1_M$). More precisely, we have a natural transformation of functors $S^{-1}(-) \rightarrow T^{-1}(-)$, and it commutes (in an appropriate sense) with the natural transformations in (1.3.4) and (1.3.5).

(1.4.3) If every element of T divides an element of S , then the transformation $\rho^{S,T}$ above is an isomorphism. Call S *saturated* if it contains all divisors of its elements. Then we can replace S by its saturation T and get essentially the same functor $S^{-1}(-) \cong T^{-1}(-)$.

(1.4.4) Given three multiplicative sets $S \subseteq T \subseteq U \subseteq A$, we have $\rho^{S,U} = \rho^{T,U} \circ \rho^{S,T}$.

(1.4.5) Given a filtered direct system of multiplicative sets $S_\alpha \subseteq A$, and S their union, there are canonical isomorphisms

$$\varinjlim S_\alpha^{-1}A \cong S^{-1}A, \quad \varinjlim S_\alpha^{-1}M \cong S^{-1}M,$$

the second one giving a natural isomorphism of functors.

(1.4.6) If S_1 and S_2 are multiplicative, then so is S_1S_2 , and one has a natural isomorphism

$$S_1^{-1}(S_2^{-1}M) \cong (S_1S_2)^{-1}M$$

such that $(m/s)/t \leftrightarrow m/st$.

1.5 Change of rings.

(1.5.1) Given a ring homomorphism $\phi: A' \rightarrow A$ and multiplicative sets $S' \subseteq A'$, $S \subseteq A$ such that $\phi(S') \subseteq S$, one has by (1.2.4) a unique $\phi^{S'}: S'^{-1}A' \rightarrow S^{-1}A$ making a commutative diagram

$$\begin{array}{ccc} S'^{-1}A' & \xrightarrow{\phi^{S'}} & S^{-1}A \\ \uparrow & & \uparrow \\ A' & \xrightarrow{\phi} & A. \end{array}$$

If $\phi(S') = S$, then $\phi^{S'}$ is *surjective*.

(1.5.2) In the above setting, if M is an A -module, it is also an A' module, and $S^{-1}M$ is also an $S'^{-1}A'$ -module. There is a canonical homomorphism of $S'^{-1}A'$ -modules

$$\sigma: S'^{-1}M \rightarrow S^{-1}M$$

sending m/s' to $m/\phi(s')$. This gives a natural transformation of functors, and if $\phi(S') = S$, it is an isomorphism. (1.4.1) is the special case $A' = A$. When $M = A$, $S'^{-1}A$ is a ring and $\sigma: S'^{-1}A \rightarrow S^{-1}A$ is a homomorphism of $S'^{-1}A'$ -algebras.

(1.5.3) Composing the maps σ with those in (1.3.4) and (1.3.5) gives natural transformations

$$\begin{aligned} (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) &\leftarrow S'^{-1}(M \otimes_A N), \\ S'^{-1} \text{Hom}_A(M, N) &\rightarrow \text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \end{aligned}$$

of which the first is an isomorphism if $\phi(S') = S$ and the second is an isomorphism if $\phi(S') = S$ and M is finitely presented.

(1.5.4-5) The functor $- \otimes_{A'} A$ from A' -modules to A modules is called *extension of scalars* (it is left adjoint to the functor in the opposite direction that sends an A -module M to M considered as an A' -module). Given A' -modules M', N' , there are natural isomorphisms of $S^{-1}A$ -modules

$$\begin{aligned} S^{-1}(N' \otimes_{A'} A) &\cong (S'^{-1}N') \otimes_{S'^{-1}A} S^{-1}A, \\ S^{-1}(M' \otimes_{A'} N' \otimes_{A'} A) &\cong (S'^{-1}M') \otimes_{S'^{-1}A} (S'^{-1}N') \otimes_{S'^{-1}A} S^{-1}A. \end{aligned}$$

and a natural homomorphism

$$S^{-1}(\text{Hom}_{A'}(M', N') \otimes_{A'} A) \rightarrow \text{Hom}_{S'^{-1}A'}(S'^{-1}M', S'^{-1}N') \otimes_{S'^{-1}A'} S^{-1}A,$$

which is an isomorphism if M' is finitely presented. The last two of these follow from the first and (1.3.4-5).

(1.5.6) In the setting of (1.5.1), suppose we also have multiplicative sets $T' \supseteq S'$, $T \supseteq S$ such that $\phi(T') \subseteq T$. Then the maps in (1.5.1-3) above are compatible with the maps $\rho^{S,T}$ in (1.4.1-2), in the sense that the obvious diagrams commute.

1.6 Identification of the module M_f with a direct limit. (1.6.1) Given $f \in A$ and an A -module M , consider the directed system

$$M_1 \xrightarrow{f} M_2 \xrightarrow{f} \cdots,$$

where every $M_n = M$, and the arrows are $m \mapsto fm$. There is a natural isomorphism

$$M_f \cong \varinjlim M_n.$$

Given another element $g \in A$, we have a homomorphism of directed systems

$$\begin{array}{ccccccc} M_1 & \xrightarrow{f} & M_2 & \xrightarrow{f} & \cdots & \xrightarrow{f} & M_n & \xrightarrow{f} & \cdots \\ 1 \downarrow & & g \downarrow & & & & g^n \downarrow & & \\ M'_1 & \xrightarrow{fg} & M'_2 & \xrightarrow{fg} & \cdots & \xrightarrow{fg} & M'_n & \xrightarrow{fg} & \cdots \end{array}$$

where all the modules here are equal to M , and the induced map from $M_f = \varinjlim M_n$ to $M_{fg} = \varinjlim M'_n$ coincides with the map $\rho^{f,fg}$ given by (1.4.1) and (1.4.3) [note that S_f is contained in the saturation of S_{fg}].

1.7 Support of a module.

(1.7.1) The *support* $\text{Supp}(M)$ of an A -module M is the set of prime ideals $\mathfrak{p} \subseteq A$ such that $M_{\mathfrak{p}} \neq 0$. If $\text{Supp}(M) = \emptyset$, then $M = 0$ [Liu, 1.2.12].

(1.7.2) If $0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$ is an exact sequence, then $\text{Supp}(M) = \text{Supp}(N) \cup \text{Supp}(P)$, by the exactness of localization.

(1.7.3) If $M = \sum_{\alpha} M_{\alpha}$ is the sum of a collection of its submodules, then $M_{\mathfrak{p}} = \sum_{\alpha} (M_{\alpha})_{\mathfrak{p}}$ by (1.3.2-3), hence $\text{Supp}(M) = \bigcup_{\alpha} \text{Supp}(M_{\alpha})$.

(1.7.4) If M is finitely generated, then $\text{Supp}(M)$ is equal to the set of primes containing the annihilator of M .

(1.7.5) If M and N are finitely generated, then $\text{Supp}(M \otimes_A N) = \text{Supp}(M) \cap \text{Supp}(N)$.