DERIVED CATEGORY AND DERIVED FUNCTORS

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References

You might want to consult these general references for more information:

1. R. Hartshorne, *Residues and Duality*, Springer Lecture Notes 20 (1966), is a standard reference, but a bit dated.
2. C. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics 38 (1994), has a useful chapter at the end on derived categories and functors.
3. B. Keller, Derived categories and their uses, in *Handbook of Algebra, Vol. 1*, M. Hazewinkel, ed., Elsevier (1996), is another helpful synopsis.
4. J.-L. Verdier's thesis *Catégories dérivées* is the original reference; also his essay with the same title in *SGA 4-1/2*, Springer Lecture Notes 569 (1977).

The presentation here incorporates additional material from the following references:

5. P. Deligne, Cohomologie à supports propres, *SGA 4*, Springer Lecture Notes 305 (1973)
6. P. Deligne, Cohomologie à support propres et construction du foncteur $f^!$, appendix to *Residues and Duality* [1].
7. J.-L. Verdier, Base change for twisted inverse image of coherent sheaves, in *Algebraic Geometry, Bombay Colloquium 1968*, Oxford Univ. Press (1969)
8. N. Spaltenstein, Resolutions of unbounded complexes, *Compositio Math.* 65 (1988) 121–154
9. A. Neeman, Grothendieck duality via Bousfield's techniques and Brown representability, *J.A.M.S.* 9, no. 1 (1996) 205–236.

1. Basic concepts

Definition 1.1. An *additive category* is a category $\mathcal{A}$ in which $\mathrm{Hom}(A, B)$ is an abelian group for all objects A, B, composition of arrows is bi-linear, and $\mathcal{A}$ has (finite) direct sums and a zero object. An *abelian category* is an additive category in which every arrow f has a kernel, cokernel, image and coimage, and the canonical map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.

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The abelian categories of interest to us will be the category of modules over a ring (including the category of abelian groups, as $\mathbb{Z}$-modules), the category of sheaves of $\mathcal{O}_X$-modules on a ringed space X, and certain abelian subcategories, as for instance of quasi-coherent $\mathcal{O}_X$-modules. It is convenient to reason in terms of elements of objects A in any of these categories (elements being sections or germs, in the case of sheaves). Arguments using elements can also be justified for abstract abelian categories, by regarding $\operatorname{Hom}(T, A)$ as the set of "T-valued elements" of an object A, much as we regard morphisms of schemes $T \to X$ as T-valued points of X.

Definition 1.2. A *complex* in an abelian category $\mathcal{A}$ is a sequence $A^\bullet$ of objects and maps (called *differentials*)

$$\cdots \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots$$

such that $d^{i+1} \circ d^i = 0$ for all i. A homomorphism $f\colon A^\bullet \to B^\bullet$ of complexes consists of maps $f^i : A^i \to B^i$ commuting with the differentials. The complexes in $\mathcal{A}$ form an abelian category $\mathbf{C}(\mathcal{A})$. The object $H^i(A^\bullet) = \ker(d^i)/\operatorname{im}(d^{i-1})$ is the i-th *cohomology* of $A^\bullet$. I'll stick to cohomology indexing. It is also conventional sometimes to use "homology" indexing defined by $A_i = A^{-i}$, $H_i(A_\bullet) = H^{-i}(A^\bullet)$.

The *shift* $A[n]^\bullet$ of $A^\bullet$ is the complex with terms $A[n]^i = A^{i+n}$ and differentials $d^i_{A[n]} = (-1)^n d^{i+n}_A$ (this sign rule provides compatibility with double complexes and mapping cones). An object A of $\mathcal{A}$ can be identified with the complex $A^0 = A$, $A^i = 0$ for $i \neq 0$. Then $A[n]$ is the complex which is A in degree $-n$.

Definition 1.3. A homomorphism of complexes $f\colon A^\bullet \to B^\bullet$ is a *quasi-isomorphism* ("qis" for short) if f induces isomorphisms $H^i(A^\bullet) \xrightarrow{\cong} H^i(B^\bullet)$ for all i. A complex is quasi-isomorphic to zero iff $H^i(A^\bullet) = 0$ for all i, or in other words, $A^\bullet$ is *acyclic.*

Definition 1.4. To a complex of complexes $(A^\bullet)^\bullet$

$$\cdots \xrightarrow{d^{-1}} (A^0)^\bullet \xrightarrow{d^0} (A^1)^\bullet \xrightarrow{d^1} (A^2)^\bullet \xrightarrow{d^2} \cdots$$

is associated a *double complex* $A^{\bullet\bullet}$ whose rows are $(A^p)^\bullet$, with differentials $(-1)^p d^\bullet_{A^p}$, and whose columns are the q-th terms $(A^\bullet)^q$, with differentials $(d^\bullet)^q$. Its *total complex* $\operatorname{Tot}(A^{\bullet\bullet})$ is defined by

$$\operatorname{Tot}(A^{\bullet\bullet})^n = \bigoplus_{p+q=n} A^{p,q},$$

with differentials given by the sum of the row and column differentials. You can verify that this in fact makes $\operatorname{Tot}(A^{\bullet\bullet})$ a complex. If the column differentials are zero, *i.e.* the differentials d^i in our initial complex of complexes $(A^\bullet)^\bullet$ are all zero, then $\operatorname{Tot}(A^{\bullet\bullet}) = \bigoplus_k (A^k)[-k]$.

Definition 1.5. Given a homomorphism $f\colon A^\bullet \to B^\bullet$, form the complex of complexes which is $A^\bullet$ in degree -1, $B^\bullet$ in degree 0, and zero in other degrees, with differential $d^{-1} = f$. The total complex of its double complex is the *mapping cone of* f, denoted $C(f)$. Explicitly,

$C(f)$ looks like this:

$$\begin{tikzcd}
{} \arrow[r, "(-)"] \arrow[ddr, "f^{-1}"'] & A^0 \arrow[r, "(-)"] \arrow[ddr, "f^0"'] & A^1 \arrow[r, "(-)"] \arrow[ddr, "f^1"'] & A^2 \arrow[r, "(-)"] \arrow[ddr, "f^2"'] & {} \\
 & \oplus & \oplus & \oplus & \\
{} \arrow[r] & B^{-1} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & {} \\
 & (-1) & (0) & (1) &
\end{tikzcd},$$

displayed so that the columns are the terms of $C(f)$.

Proposition 1.6. *(i) $C(f)$ is functorial in the triple $A \xrightarrow{f} B$.*

(ii) There is a canonical exact sequence of complexes

$$0 \to B \xrightarrow{i} C(f) \xrightarrow{p} A[1] \to 0.$$

(iii) Let $K = \ker(f)$, $Q = \operatorname{coker}(f)$. The canonical maps $K \hookrightarrow A$ and $B \to Q$ factor through the maps i, p in (ii), as

$$K \overset{k}{\hookrightarrow} C(f)[-1] \xrightarrow{p[-1]} A, \quad B \xrightarrow{i} C(f) \xrightarrow{q} Q.$$

(iv) If f is injective, then $q\colon C(f) \underset{qis}{\xrightarrow{\cong}} Q$ is a quasi-isomorphism.

(v) If f is surjective, then $k[1]\colon K[1] \underset{qis}{\xrightarrow{\cong}} C(f)$ is a quasi-isomorphism.

(vi) If f is bijective, then $C(f)$ is acyclic.

(vii) For each i, the exact sequence in (ii) induces an exact sequence

$$H^i(B) \to H^i(C(f)) \to H^{i+1}(A).$$

Proof. Exercise. □

Definition 1.7. A homomorphism of complexes $f\colon A^\bullet \to B^\bullet$ is *null-homotopic*, written $f \sim 0$, if there exist maps $s^i\colon A^i \to B^{i-1}$ such that $f^i = d_B^{i-1}s^i + s^{i+1}d_A^i$ for all i. Two homomorphisms f, g are *homotopic*, written $f \sim g$, if $f - g$ is null-homotopic. If f is null-homotopic, then so is every $h \circ f$ and $f \circ j$. Hence there is a well defined *homotopy category* $\mathbf{K}(\mathcal{A})$ whose objects are complexes, and whose arrows are homotopy classes of homomorphisms in $\mathbf{C}(\mathcal{A})$.

Remark 1.8. There is a complex $\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)$ defined by

$$\operatorname{Hom}^i(A^\bullet, B^\bullet) = \prod_j \operatorname{Hom}(A^j, B^{j+i}),$$

with differentials $(d^i(\phi))^j = d_B^{j+i} \circ \phi^j - (-1)^i \phi^{j+1} \circ d_A^j$ for $\phi = (\phi^j\colon A^j \to B^{j+i}) \in \operatorname{Hom}^i(A^\bullet, B^\bullet)$. Note that $f \in \operatorname{Hom}^0(A^\bullet, B^\bullet)$ is a cycle, *i.e.*, $d^0(f) = 0$, iff f is a homomorphism of complexes, and f is a boundary, *i.e.*, $f \in \operatorname{im}(d^{-1})$, iff f is null-homotopic. Thus $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A^\bullet, B^\bullet) \cong H^0(\operatorname{Hom}^\bullet(A^\bullet, B^\bullet))$.

Proposition 1.9. *The following properties of a homomorphism* $f\colon A \to B$ *in* $\mathbf{C}(\mathcal{A})$ *are equivalent:*

(a) $f \sim 0$.

(b) f *factors through the canonical map* $i\colon A \to C(1_A)$ *given by 1.6(ii) for* 1_A.

(c) f *factors through the canonical map* $p[-1]\colon C(1_B)[-1] \to B$ *given by 1.6(ii) for* 1_B.

(d) The exact sequence 1.6(ii) for f *splits.*

Proof. Exercise. □

Corollary 1.10. *Homotopic maps* $f \sim g$ *induce the same maps on cohomology. In particular, the cohomology functors* $\mathbf{K}(\mathcal{A}) \to \mathcal{A}$, $A \mapsto H^i(A)$ *are well-defined.*

Proof. A null-homotopic map $f\colon A \to B$ induces the zero map on cohomology because it factors through $C(1_A)$, which is acyclic by 1.6(vi). □

Corollary 1.11. *Every homotopy equivalence (i.e., every homomorphism invertible in* $\mathbf{K}(\mathcal{A})$*) is a quasi-isomorphism.*

Remark 1.12. Proposition 1.6(vi) can be strengthened (exercise) to say that if f is bijective, then $C(f)$ is homotopy-equivalent to zero. However, it is not true that every acyclic complex is homotopy-equivalent to zero, nor do 1.6(iv, v) hold for homotopy-equivalence.

2. TRIANGLES

The homotopy category $\mathbf{K}(\mathcal{A})$ and the derived category $\mathbf{D}(\mathcal{A})$, to be introduced in §3, are additive but not abelian categories. Instead, they share an extra structure described by a distinguished collection of *exact triangles*. Although we are mainly interested in the derived category, we first consider triangles in the homotopy category. It will be easier to deduce the main properties of the derived category after this intermediate step.

Lemma 2.1. *Given a homomorphism of complexes* $f\colon A \to B$*, each composite of two successive maps in the sequence*

$$\cdots \to A \xrightarrow{f} B \xrightarrow{i} C(f) \xrightarrow{p} A[1] \xrightarrow{f[1]} B[1] \to \cdots$$

induced by 1.6(ii) is zero in $\mathbf{K}(\mathcal{A})$.

Proof. The composite $B \to C(f) \to A[1]$ is already zero in $\mathbf{C}(\mathcal{A})$. For $A \xrightarrow{f} B \xrightarrow{i} C(f)$, the diagram

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

yields a map $C(1_A) \to C(f)$, and one checks easily that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i(1_A)"'] & B \arrow[d, "i"'] \\
C(1_A) \arrow[r] & C(f)
\end{tikzcd}$$

commutes. Hence $i \circ f$ is null-homotopic, by 1.9(b). A similar argument takes care of $C(f) \xrightarrow{p} A[1] \xrightarrow{f[1]} B[1]$, and the rest follows by shift-invariance. □

Definition 2.2. A *triangle* in an additive category with shift functors $A \mapsto A[n]$ is a sequence

$$A \to B \to C \to A[1]$$

for which the conclusion of 2.1 holds (with C in place of $C(f)$). A *morphism of triangles* is a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "u"'] & B \arrow[r] \arrow[d, "v"'] & C \arrow[r] \arrow[d, "w"'] & A[1] \arrow[d, "{u[1]}"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}.$$

A *standard triangle* in $\mathbf{K}(\mathcal{A})$ is a triangle of the form

$$A \xrightarrow{f} B \xrightarrow{i} C(f) \xrightarrow{p} A[1],$$

induced from a homomorphism $f\colon A \to B$ by 1.6(ii). An *exact triangle* in $\mathbf{K}(\mathcal{A})$ is a triangle isomorphic to a standard triangle.

Triangles are also displayed like this:

$$\begin{tikzcd}
 & C \arrow[dl, "+1"'] & \\
A \arrow[rr] & & B. \arrow[ul]
\end{tikzcd}$$

Proposition 2.3. *Exact triangles in* $\mathbf{K}(\mathcal{A})$ *satisfy the following axioms:*

(o) Any triangle isomorphic to an exact triangle is exact.

(i) Every arrow $f\colon A \to B$ *is the base of an exact triangle* $A \xrightarrow{f} B \to C \to A[1]$*. For every object* A*, the triangle* $A \xrightarrow{1_A} A \to 0 \to A[1]$ *is exact.*

(ii) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ *is an exact triangle, then so are its "rotations"*

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1], \quad C[-1] \xrightarrow{-h[-1]} A \xrightarrow{f} B \xrightarrow{g} C.$$

(iii) Given exact triangles $A \xrightarrow{f} B \to C \to A[1]$ *and* $A' \xrightarrow{g} B' \to C' \to A'[1]$, *every commutative diagram in* $\mathbf{K}(\mathcal{A})$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "h'"'] \\
A' \arrow[r, "g"] & B'
\end{tikzcd}$$

extends to a morphism of triangles

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d] \\
A' \arrow[r, "g"] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}.$$

(iv) A direct sum of exact triangles is exact.

Proof. Axioms (o), (iv) and the first part of (i) are obvious. For the second part of (i), we have $C(1_A) \cong 0$ in $\mathbf{K}(\mathcal{A})$ by Remark 1.12.

For (ii), using shift invariance (see remark below), it suffices to verify the first rotation. We can assume the given triangle is standard, $A \xrightarrow{f} B \xrightarrow{i} C(f) \xrightarrow{p} A[1]$. Then we must show that $A[1] \cong C(i)$ in $\mathbf{K}(\mathcal{A})$, via an isomorphism such that the composite $A[1] \to C(i) \xrightarrow{p(i)} B[1]$ is $-f[1]$ and $C(f) \xrightarrow{i(i)} C(i) \to A[1]$ is p. Now, $C(i)$ is identical to the mapping cone of the map $h\colon A \to C(1_B)$ obtained by composing $i(1_B)\colon B \to C(1_B)$ with f. This gives a canonical map $\pi = p(h) : C(i) \to A[1]$, by 1.6(ii). But $C(i)$ is also identical to the mapping cone of $(f, 1_B)\colon A \oplus B \to B$, whose kernel is isomorphic to A. This gives a map $\iota = k[1] : A[1] \to C(i)$, by 1.6(iii). One checks that $\pi \circ \iota = 1_{A[1]}$, $p(i) \circ \iota = -f[1]$, $\pi \circ i(i) = p$, and $\iota \circ \pi \sim 1_{C(i)}$.

For (iii), we can assume both triangles are standard. If the given diagram commutes up to a homotopy $s\colon gh \sim h'f$, you can check that $(a^{i+1}, b^i) \mapsto (h(a^{i+1}), h'(b^i) + s(a^{i+1}))$ is a homomorphism $C(f) \to C(g)$ that yields the desired morphism of triangles in $\mathbf{K}(\mathcal{A})$. ☐

Remarks 2.4. (a) *Warning:* The morphism of triangles in (iii) is not canonical, but depends on the choice of an isomorphism between each of the given triangles and a standard triangle. Thus we do *not* have a functorial "mapping-cone" construction assigning to each arrow $f\colon A \to B$ in $\mathbf{K}(\mathcal{A})$ an exact triangle with f as its base.

(b) Triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ and $A \xrightarrow{-f} B \xrightarrow{-g} C \xrightarrow{h} A[1]$ are isomorphic via -1_B, and likewise if we change any two signs. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is exact, a triangle such as $A \xrightarrow{-f} B \xrightarrow{-g} C \xrightarrow{-h} A[1]$ with one or three signs changed is *anti-exact*. Note that $C(f)[1] = C(-f[1])$, so the shift $A[1] \xrightarrow{f[1]} B[1] \xrightarrow{g[1]} C[1] \xrightarrow{h[1]} A[1]$ is anti-exact, while $A[1] \xrightarrow{-f[1]} B[1] \xrightarrow{-g[1]} C[1] \xrightarrow{-h[1]} A[1]$, gotten by rotating our original triangle three times, is exact.

(c) Verdier defined a *triangulated category* to be an additive category with shift functors, satisfying axioms 2.3(o–iii) and an additional, more complicated, "octahedral axiom" which relates exact triangles based on f, g and $g \circ f$, and implies (iv). The fundamental examples of triangulated categories in the sense of Verdier are homotopy categories of complexes, derived categories, and the stable homotopy category of spectra of CW-complexes in topology. The logical significance of the octahedral axiom remains a bit murky. On one hand, it is not needed at all in the elementary applications of derived categories to supplying a good framework for homological algebra and sheaf cohomology. On the other hand, deeper aspects of the theory might some day involve properties which are enjoyed by the natural examples of triangulated categories, but are stronger than Verdier's axioms.

3. The derived category

Definition 3.1. The *derived category* $\mathbf{D}(\mathcal{A}) = \mathbf{C}(\mathcal{A})[Q^{-1}]$ of $\mathcal{A}$ is the category obtained from $\mathbf{C}(\mathcal{A})$ by formally inverting all quasi-isomorphisms. More precisely, $\mathbf{D}(\mathcal{A})$ is equipped with a tautological functor $j : \mathbf{C}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$, and has the universal property that any functor $F \colon \mathbf{C}(\mathcal{A}) \to \mathcal{B}$ factors (uniquely) through $\mathbf{D}(\mathcal{A})$ iff F sends all quasi-isomorphisms to isomorphisms in $\mathcal{B}$.

Remark 3.2. Some set-theoretic foundational issues in category theory impinge on the construction of derived categories. The objects of an ordinary category $\mathcal{C}$ need not form a set, but $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is required to be a set for all objects A, B of $\mathcal{C}$. A category whose objects do form a set is called *small.* Conversely, we may allow each $\operatorname{Hom}_{\mathcal{C}}(A, B)$ to be a proper class, in which case $\mathcal{C}$ is called *large.*

If $\mathcal{C}$ is a small category, we can formally invert any subset Q of its arrows to get another small category $\mathcal{C}[Q^{-1}]$ with the same objects as $\mathcal{C}$, the arrows of $\mathcal{C}[Q^{-1}]$ being defined by suitable generators and relations. If $\mathcal{C}$ is an ordinary category, we can again construct $\mathcal{C}[Q^{-1}]$, but in general only as a large category.

In practice there are several strategies for coping with the above difficulties. (1) Ignore them—as I will do in these notes. (2) Fix a "universal" set U of some large cardinality and admit only small categories contained in U. This approach is perfectly satisfactory for all concrete problems of algebraic geometry. (3) Use Gödel-Bernays set theory, which provides explicitly for a hierarchy of proper classes beyond sets. (4) Prove that particular derived categories of interest are equivalent to ordinary categories. For example, this is the case for the bounded-below derived category $\mathbf{D}^{+}(\mathcal{A})$ if $\mathcal{A}$ is an abelian category with enough injectives. The otherwise useful textbook by Weibel contains some nonsense about "proving that the derived category exists in our universe," which appears to be a garbled reference to this last strategy.

In any event, the genuine mathematical issues involved do not concern set-theoretic technicalities, but rather concrete questions of how to describe an arrow in $\mathbf{D}(\mathcal{A})$, and how to recognize when two arrows are equal.

By the definition of quasi-isomorphism, the cohomology functors $H^i \colon \mathbf{C}(\mathcal{A}) \to \mathcal{A}$ factor through unique functors $H^i \colon \mathbf{D}(\mathcal{A}) \to \mathcal{A}$. If A is an object of $\mathcal{A}$, let $A[0]$ denote the complex which is A in degree zero, and 0 in other degrees. Then $H^0(A[0]) = A$, so $A \mapsto A[0]$ is a fully faithful embedding of $\mathcal{A}$ into $\mathbf{D}(\mathcal{A})$, with left inverse given by the functor H^0. Usually we just identify A with $A[0]$ and regard $\mathcal{A}$ as a full subcategory of $\mathbf{D}(\mathcal{A})$.

Proposition 3.3. *The canonical functor* $\mathbf{C}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ *factors (uniquely) through* $\mathbf{K}(\mathcal{A})$.

Proof. It suffices to prove that $f \sim 0$ implies $f = 0$ in $\mathbf{D}(\mathcal{A})$. This is immediate from 1.9(b), since $C(1_A) \cong 0$ in $\mathbf{D}(\mathcal{A})$, by 1.6(vi). ☐

Corollary 3.4. *The derived category* $\mathbf{D}(\mathcal{A})$ *can also be identified with* $\mathbf{K}(\mathcal{A})[Q^{-1}]$.

Remarks 3.5. (a) Traditionally, $\mathbf{D}(\mathcal{A})$ is often defined as $\mathbf{K}(\mathcal{A})[Q^{-1}]$. This tends to overemphasize the role of the homotopy category, which is not essential to the definition, although it is a useful auxiliary device for understanding many properties of $\mathbf{D}(\mathcal{A})$.

(b) Equality in $\mathbf{D}(\mathcal{A})$ of homomorphisms $f, g \in \operatorname{Hom}_{\mathbf{C}(\mathcal{A})}(A, B)$ does *not* imply that f and g are homotopic. A criterion for equality of arrows in the derived category is given by 3.22(ii), below.

Definition 3.6. An *exact triangle* in $\mathbf{D}(\mathcal{A})$ is a triangle isomorphic in $\mathbf{D}(\mathcal{A})$ to a standard triangle, as in 2.2. Equivalently (by 1.11), a triangle in $\mathbf{D}(\mathcal{A})$ is exact iff it is isomorphic in $\mathbf{D}(\mathcal{A})$ to an exact triangle of $\mathbf{K}(\mathcal{A})$.

An advantage of the derived category is that every exact sequence in $\mathbf{C}(\mathcal{A})$ gives rise to an exact triangle in $\mathbf{D}(\mathcal{A})$, which is not the case in $\mathbf{K}(\mathcal{A})$.

Proposition 3.7. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence in* $\mathbf{C}(\mathcal{A})$. *Then the diagram*

$$\begin{tikzcd}
C(f) \arrow[r, "\cong\, q(f)", "qis"'] \arrow[d, "p(f)"'] & C \arrow[d, "i(g)"'] \\
A[1] \arrow[r, "\cong\, k(g)[1]", "qis"'] & C(g)
\end{tikzcd}$$

anti-commutes in $\mathbf{K}(\mathcal{A})$, *and hence in* $\mathbf{D}(\mathcal{A})$.

Proof. The formula for $k(g)[1] \circ p(f)$ is $(a^{i+1}, b^i) \mapsto (f(a^{i+1}), 0)$, and for $i(g) \circ q(f)$ it is $(a^{i+1}, b^i) \mapsto (0, g(b^i))$. Then $s^i(a^{i+1}, b^i) = (b^i, 0)$ is a homotopy between $k(g)[1] \circ p(f)$ and $-i(g) \circ q(f)$. ☐

Definition 3.8. The map $h\colon C \to A[1]$ in $\mathbf{D}(\mathcal{A})$, given in terms of the diagram in 3.7 by $h = p(f) \circ q(f)^{-1} = -k(g)[1]^{-1} \circ i(g)$, is the *connecting homomorphism* of the exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$.

Proposition 3.9. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence in* $\mathbf{C}(\mathcal{A})$. *There is an exact triangle*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1],$$

in $\mathbf{D}(\mathcal{A})$*, where* h *is the connecting homomorphism.*

Proof. By 1.6(iv), $q(f)\colon C(f) \to C$ is a quasi-isomorphism whose composite with the canonical map $i\colon B \to C(f)$ is g. Hence the standard triangle based on f is isomorphic in $\mathbf{D}(\mathcal{A})$ to the triangle above. ☐

Lemma 3.10. *Exact triangles in* $\mathbf{D}(\mathcal{A})$ *satisfy the rotation axiom 2.3(ii).*

This is obvious from the definition and 2.3(ii) for $\mathbf{K}(\mathcal{A})$. Below we will see that in fact all the axioms 2.3(o–iv) hold in $\mathbf{D}(\mathcal{A})$. First we need two preliminaries: the cohomology long exact sequence, which is a basic tool for all of homological algebra, and a description of the arrows in $\mathbf{D}(\mathcal{A})$.

Proposition 3.11. *If* $A \to B \to C \to A[1]$ *is an exact triangle in* $\mathbf{D}(\mathcal{A})$*—in particular, if* $0 \to A \to B \to C \to 0$ *is an exact sequence of complexes and* $h\colon C \to A[1]$ *is the connecting homomorphism in* $\mathbf{D}(\mathcal{A})$*—there is an induced long exact sequence of cohomology groups*

$$\cdots \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to \cdots .$$

Proof. Apply 3.10 and 1.6(vii). ☐

Corollary 3.12. *If a morphism between exact triangles in* $\mathbf{D}(\mathcal{A})$ *is an isomorphism at two of the three corners of the triangle, then it is an isomorphism of triangles.*

Proof. An arrow in $\mathbf{D}(\mathcal{A})$ is an isomorphism iff it induces isomorphisms in cohomology. In the diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(A) \arrow[r] \arrow[d] & H^i(B) \arrow[r] \arrow[d] & H^i(C) \arrow[r] \arrow[d] & H^{i+1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^i(A') \arrow[r] & H^i(B') \arrow[r] & H^i(C) \arrow[r] & H^{i+1}(A') \arrow[r] & \cdots
\end{tikzcd},$$

every third vertical arrow is an isomorphism, given that the others are. ☐

Corollary 3.13. *Let* $A \xrightarrow{f} B \to C \to A[1]$ *be an exact triangle (in either* $\mathbf{D}(\mathcal{A})$ *or* $\mathbf{K}(\mathcal{A})$*) based on a homomorphism* $f\colon A \to B$ *in* $\mathbf{C}(\mathcal{A})$*. Then* f *is a quasi-isomorphism if and only if* C *is acyclic.*

Proposition 3.14. *If* $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ *is an exact triangle in* $\mathbf{K}(\mathcal{A})$*, then for any complex* X*, there are induced long exact sequences*

$$\cdots \to \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, A) \to \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, B) \to \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, C) \to \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, A[1]) \to \cdots$$
$$\cdots \leftarrow \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, X) \leftarrow \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(B, X) \leftarrow \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(C, X) \leftarrow \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A[1], X) \leftarrow \cdots$$

Proof. We prove the second sequence; a similar argument applies to the first. Suppose $f\colon B \to X$ satisfies $fu \sim 0$. From 2.3(i,iii) we get a morphism of triangles

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d] & B \arrow[r, "v"] \arrow[d, "f"'] & C \arrow[r, "w"] \arrow[d, "g"'] & A[1] \arrow[d] \\
0 \arrow[r] & X \arrow[r, "1_X"] & X \arrow[r] & 0
\end{tikzcd},$$

which shows $f \sim gv$ for some $g\colon C \to X$. In other words, $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, X) \leftarrow \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(B, X) \leftarrow \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(C, X)$ is exact, and the rest follows by the rotation axiom, 2.3(ii). ☐

Corollary 3.15. *Corollary 3.12 also holds in* $\mathbf{K}(\mathcal{A})$*, and indeed in any "weakly" triangulated category, satisfying axioms 2.3(o-iv).*

Proof. In the proof of 3.12 we can use either long exact sequence in 3.14 in place of the one in 3.11, together with the fact that the functor $\mathrm{Hom}(-, C)$ (resp. $\mathrm{Hom}(C, -)$) determines C up to canonical isomorphism. ☐

Remark 3.16. The proofs show that 3.12 and 3.14 hold in any weakly triangulated category.

Our next goal is to describe the arrows in $\mathbf{D}(\mathcal{A})$.

Definition 3.17. A class of Q of arrows in a category $\mathcal{C}$ is a (left) *Ore system* if it satisfies the following conditions:

(a) Q is multiplicative, *i.e.* $Q \circ Q \subseteq Q$ and $1_X \in Q$ for every object X of $\mathcal{C}$.

(b) Every pair of arrows $A' \xleftarrow{q} A \xrightarrow{f} B$ with $q \in Q$ can be completed to a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[d, "r"'] \\
A' \arrow[r, "g"] & B'
\end{tikzcd}$$

with $r \in Q$.

(c) If $f \circ q = 0$ with $q \in Q$, there exists $r \in Q$ such that $r \circ f = 0$.

A *right* Ore system is defined dually.

Remark 3.18. A category is *filtered* if every pair $f : A \to B$, $f' : A \to B'$ of morphisms from the same object A can be completed to a commutative square. An inductive system of sets $(X_i)_{i \in I}$ indexed by a filtered category I is also said to be *filtered*. A filtered inductive system has the property that elements $x \in X_i$, $x' \in X_{i'}$ represent the same element of the direct limit $\varinjlim(X_i)$ if and only if there exist arrows $\alpha\colon X_i \to X_j$, $\alpha'\colon X_{i'} \to X_j$ in I such that $\alpha(x) = \alpha'(x')$. To see this, one checks that in a filtered inductive system, the preceding condition defines an equivalence relation $x \equiv x'$, and then $\varinjlim(X_i) = (\bigsqcup_{i \in I} X_i)/\equiv$.

Let Q be a left Ore system in $\mathcal{C}$, fix an object A in $\mathcal{C}$, and let $Q\backslash A$ be the category of arrows $A \xrightarrow{q} A'$, where $q \in Q$, with morphisms the commutative triangles

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[dr, "q'"'] & A' \arrow[d] \\
 & A''
\end{tikzcd} .$$

Then (a–c) imply (exercise) that $Q\backslash A$ is a filtered category.

Proposition 3.19. *Assume Q is a left Ore system in $\mathcal{C}$. Then morphisms in $\mathcal{C}[Q^{-1}]$ are given by the filtered direct limits*

$$\operatorname{Hom}_{\mathcal{C}[Q^{-1}]}(A,B) = \varinjlim_{B \xrightarrow[q]{} B'} \operatorname{Hom}_{\mathcal{C}}(A,B').$$

Denoting an element $A \xrightarrow{f} B' \xleftarrow{q} B$ of this direct limit by $q^{-1}f$, the composition law is given by $(s^{-1}f)\circ(q^{-1}h) = (rs)^{-1}(gh)$, where $B' \xleftarrow{q} B \xrightarrow{f} C'$ completes as in (b) to a diagram such that $gq = rf$.

Proof. Using 3.17 (c), one verifies that any two diagram completions $A' \xrightarrow{g} B' \xleftarrow{r} B$, $A' \xrightarrow{g'} B'' \xleftarrow{r'} B$ in (b) represent the same element $r^{-1}g = r'^{-1}g'$ of $\lim_{B\to B'} \operatorname{Hom}_{\mathcal{C}}(A',B')$. This given, we can define a category $\mathcal{C}'$ with $\operatorname{Hom}_{\mathcal{C}'}(A,B) = \lim_{B\to B'} \operatorname{Hom}_{\mathcal{C}}(A,B')$, and check that the composition law specified in the proposition is well-defined and associative. There is an obvious functor $j\colon \mathcal{C} \to \mathcal{C}'$ sending $f\colon A \to B$ to $1_B^{-1}f$, and is immediate that for any $q\colon A \to B$ in Q, $j(q)$ has inverse $q^{-1}1_B$. It is also immediate that $\mathcal{C}'$ has the universal property of $\mathcal{C}[Q^{-1}]$. Namely, given a functor $F\colon \mathcal{C} \to \mathcal{B}$ such that $F(q)$ is invertible for all $q \in Q$, F extends to $\mathcal{C}'$ by $F(q^{-1}f) = F(q)^{-1}F(f)$, which is easily seen to be independent of the choice of representative $q^{-1}f$. ☐

Lemma 3.20. *The quasi-isomorphisms form a left and right Ore system in $\mathbf{K}(\mathcal{A})$.*

Proof. By duality, "left" suffices. Condition (a) is trivial. For (b) take B' to be the mapping cone of $(q,f)\colon A \to A' \oplus B$, with $(g,r)\colon A' \oplus B \to B'$ the canonical map i in 1.6(ii). This mapping cone is identical to the mapping cone of the map $h\colon C(q)[-1] \to B$ given by composing f with the canonical map $p[-1]\colon C(q)[-1] \to A$. The map $r\colon B \to B'$ coincides under this identification with the canonical map $B \to C(h)$. By 3.13, $C(q)$ is acyclic, hence r is a quasi-isomorphism by another application of 3.13.

For (c), given $A' \xrightarrow{q} A \xrightarrow{f} B$, let $C = C(q)$, $i\colon A \to C$ the canonical map. Since $fq = 0$ in $\mathbf{K}(\mathcal{A})$, the second long exact sequence in 3.14 implies that $f = gi$ for some $g\colon C \to B$. Let $B' = C(g)$, $r\colon B \to B'$ the canonical map $i(g)$. Now, C is acyclic by 3.13, hence r is a quasi-isomorphism by 3.13 again. Finally, $rg = 0$ in $\mathbf{K}(\mathcal{A})$ by 2.1, hence $rf = rgi = 0$. ☐

Remark 3.21. The quasi-isomorphisms satisfy conditions (a) and (b) for an Ore system in $\mathbf{C}(\mathcal{A})$, but we need to work in $\mathbf{K}(\mathcal{A})$ to have (c) hold.

Corollary 3.22. *(i) Every arrow in* $\mathbf{D}(\mathcal{A})$ *factors as* $q^{-1}f$ *and as* gr^{-1}*, where* f*,* g*,* q*,* r *are homomorphisms in* $\mathbf{C}(\mathcal{A})$*, with* q*,* r *quasi-isomorphisms.*

(ii) A homomorphism $f\colon A \to B$ *in* $\mathbf{C}(\mathcal{A})$ *is zero in* $\mathbf{D}(\mathcal{A})$ *if and only if the equivalent conditions hold: (a) there exists a quasi-isomorphism* $q\colon B \to B'$ *such that* $qf \sim 0$*; (b) there exists a quasi-isomorphism* $r\colon A' \to A$ *such that* $fr \sim 0$*.*

Corollary 3.23. *The exact triangles in* $\mathbf{D}(\mathcal{A})$ *satisfy axioms 2.3(o–iv).*

Proof. Axiom (o) holds by definition, (iv) is clear, and we have already seen (ii). Axioms (i) and (iii) follow easily from the corresponding axioms in $\mathbf{K}(\mathcal{A})$, using 3.22(i). ☐

Corollary 3.24. *Proposition 3.14 also holds in* $\mathbf{D}(\mathcal{A})$*.*

Proof. See 3.16. ☐

Corollary 3.25. *The exact triangle based on an arrow* $f\colon A \to B$ *in* $\mathbf{D}(\mathcal{A})$ *is unique up to (non-canonical) isomorphism.*

Proof. Follows from axiom (iii) and 3.12. ☐

Remarks 3.26. (a) The octahedral axiom follows similarly, so $\mathbf{D}(\mathcal{A})$ is a triangulated category in the sense of Verdier.

(b) The reasoning employed above applies more generally. Let $\mathcal{K}$ be a triangulated category and $\mathcal{N} \subseteq \mathcal{K}$ a full triangulated subcategory, closed under isomorphisms in $\mathcal{K}$. Let Q consist of the arrows in $\mathcal{K}$ such that the third vertex of any exact triangle based on $q \in Q$ is an object of $\mathcal{N}$. Then Q is a left and right Ore system in $\mathcal{K}$, and $\mathcal{D} = \mathcal{K}[Q^{-1}]$ is a triangulated category, also denoted $\mathcal{D} = \mathcal{K}/\mathcal{N}$. In our case, $\mathcal{K} = \mathbf{K}(\mathcal{A})$, with $\mathcal{N}$ consisting of the acyclic complexes. By 3.13, this is equivalent to Q consisting of the quasi-isomorphisms.

4. DERIVED FUNCTORS

We will use Deligne's method [5] of defining and constructing derived functors.

Definition 4.1. Given an complex A, let $qis\backslash A$ be the category of quasi-isomorphisms $A \underset{qis}{\xrightarrow{\simeq}} A'$ in $\mathbf{K}(\mathcal{A})$, with morphisms the commutative triangles as in 3.18.

By 3.18, $qis\backslash A$ is a filtered category. Given a functor $F\colon \mathbf{K}(\mathcal{A}) \to \mathcal{C}$ and an object Y of $\mathcal{C}$, we have a filtered inductive limit of sets, functorial in Y,

$$\varinjlim_{A \underset{qis}{\xrightarrow{\simeq}} A'} \big(\mathrm{Hom}_{\mathcal{C}}(Y, F(A'))\big).$$

Proposition 4.2. *Let* $F\colon \mathbf{K}(\mathcal{A}) \to \mathcal{C}$ *be any functor. To each arrow* $f\colon A \to B$ *in* $\mathbf{K}(\mathcal{A})$ *there is canonically associated a natural transformation*

$$r_F(f)\colon \varinjlim_{A \underset{qis}{\xrightarrow{\simeq}} A'} \big(\mathrm{Hom}_{\mathcal{C}}(-, F(A'))\big) \to \varinjlim_{B \underset{qis}{\xrightarrow{\simeq}} B'} \big(\mathrm{Hom}_{\mathcal{C}}(-, F(B'))\big) \tag{1}$$

between functors from $\mathcal{C}^{\mathrm{op}}$ *to* Sets*. This gives a functor* $r_F\colon \mathbf{K}(\mathcal{A}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Sets}})$*.*

Proof. Recall (3.20) that the quasi-isomorphisms $Q \subseteq \mathbf{K}(\mathcal{A})$ form a left Ore system (3.17). Given $A \xrightarrow{q} A'$ in $qis\backslash A$, we can define a an arrow $F(A') \to F(B')$ in $\mathcal{C}$, for some B' in $qis\backslash B$, by completing the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[d, "r"'] \\ A' \arrow[r, "g"] & B' \end{tikzcd} \tag{2}$$

in $\mathbf{K}(\mathcal{A})$ and applying F to the bottom row. Equivalently, this defines a natural map $\rho_{A'}\colon \operatorname{Hom}_{\mathcal{C}}(-, F(A')) \to \operatorname{Hom}_{\mathcal{C}}(-, F(B'))$. As in the proof of 3.19, any two completed diagrams (2) factor into a third. Hence the natural map $\operatorname{Hom}_{\mathcal{C}}(-, F(A')) \to \lim_{B \underset{qis}{\xrightarrow{\simeq}} B'}\big(\operatorname{Hom}_{\mathcal{C}}(-, F(B'))\big)$ represented by $\rho_{A'}$ is independent of the choice of completion (2). Since $\rho_{A'}$ is functorial with respect to A' in $qis\backslash A$, these maps combine to give (1). One checks easily that $r_F(f)$ defined this way is functorial in f. □

Definition 4.3. Any category $\mathcal{C}$ has a fully faithful *Yoneda embedding* $\mathcal{C} \hookrightarrow \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Sets}})$ given by $X \mapsto \operatorname{Hom}_{\mathcal{C}}(-, X)$. The functor category $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Sets}})$ has direct limits, inherited from $\underline{\mathrm{Sets}}$. The closure under filtered direct limits of the image of $\mathcal{C}$ in $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Sets}})$ is called the *ind-completion* $\operatorname{Ind}(\mathcal{C})$ of $\mathcal{C}$.

The dual concept, constructed from the dual Yoneda embedding $\mathcal{C} \hookrightarrow \operatorname{Fun}(\mathcal{C}, \underline{\mathrm{Sets}})^{\mathrm{op}}$, $X \mapsto \operatorname{Hom}_{\mathcal{C}}(X, -)$, is the *pro-completion* $\operatorname{Pro}(\mathcal{C})$ of $\mathcal{C}$. (In other words, $\operatorname{Pro}(\mathcal{C})^{\mathrm{op}} = \operatorname{Ind}(\mathcal{C}^{\mathrm{op}})$.)

Remark 4.4. (a) $\operatorname{Ind}(\mathcal{C})$ has the universal property that any functor $F\colon \mathcal{C} \to \mathcal{D}$ into a category $\mathcal{D}$ with filtered direct limits extends uniquely to a functor $\widetilde{F}\colon \operatorname{Ind}(\mathcal{C}) \to \mathcal{D}$ which preserves filtered direct limits. In particular, any functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a functor $\operatorname{Ind}(F)\colon \operatorname{Ind}(\mathcal{C}) \to \operatorname{Ind}(\mathcal{D})$ commuting with the inclusions, and if $\mathcal{C}$ has filtered direct limits, the inclusion has a canonical left inverse $\operatorname{Ind}(\mathcal{C}) \to \mathcal{C}$.

(b) In terms of the Yoneda embedding, the condition for $X = \varinjlim_\lambda (A_\lambda)$ to be the limit of a filtered inductive system (A_λ) in $\mathcal{C}$ is $\operatorname{Hom}(X, -) = \varprojlim_\lambda \operatorname{Hom}(A_\lambda, -)$, which is *not* the same as X being isomorphic to the ind-object "$\varinjlim$"(A_λ) in $\operatorname{Ind}(\mathcal{C})$. The latter condition is stronger: it means that the system (A_λ) is *essentially constant with limit* X—see 4.7.

In this language, (1) defines a functor $r_F\colon \mathbf{K}(\mathcal{A}) \to \operatorname{Ind}(\mathcal{C})$. When F is the identity functor, we obtain in particular a canonical functor $j = r_{1_{\mathbf{K}(\mathcal{A})}}\colon \mathbf{K}(\mathcal{A}) \to \operatorname{Ind}(\mathbf{K}(\mathcal{A}))$ sending A to the ind-object "$\varinjlim$"$(qis\backslash A) = \lim_{A \underset{qis}{\xrightarrow{\simeq}} A'}(\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(-, A'))$. For arbitrary F, we have $r_F = \operatorname{Ind}(F) \circ j$.

Proposition 4.5. *The image of the functor* $j : \mathbf{K}(\mathcal{A}) \to \operatorname{Ind}(\mathbf{K}(\mathcal{A}))$, $A \mapsto$ "$\varinjlim$"$(qis\backslash A)$ *is isomorphic to* $\mathbf{D}(\mathcal{A})$, *with* j *corresponding to the canonical functor* $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$.

Proof. The definitions having been understood, this is merely a restatement of 3.19. □

Now let $F\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ be an *exact functor* of triangulated categories, *i.e.* an additive functor which commutes with shifts and preserves exact triangles. In particular, any additive functor $F\colon \mathcal{A} \to \mathcal{B}$ induces an exact functor $F\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ (also denoted F by abuse of notation). In practice, we only deal with functors of this last form.

Composing the canonical functor $j_{\mathcal{B}}\colon \mathbf{K}(\mathcal{B}) \to \mathbf{D}(\mathcal{B})$ with F, and applying 4.2–4.5 (with $\mathcal{C} = \mathbf{D}(\mathcal{B})$), we define a functor $RF = r_{j_{\mathcal{B}}F}\colon \mathbf{D}(\mathcal{A}) \to \operatorname{Ind}(\mathbf{D}(\mathcal{B}))$.

Definition 4.6. The *right derived functor of F is defined at A* in $\mathbf{D}(\mathcal{A})$, with *value X* in $\mathbf{D}(\mathcal{B})$, if $RF(A) = X$ belongs to $\mathbf{D}(\mathcal{B}) \subseteq \operatorname{Ind}(\mathbf{D}(\mathcal{B}))$.

It is an exercise for the reader to work out the dual definition of *left derived functor LF*.

4.7. To be more precise, the definition means that $RF(A)$ and $X \in \mathbf{D}(\mathcal{B})$ represent isomorphic functors

$$\operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(-,X) \cong \varinjlim_{\substack{A \xrightarrow{\cong} A' \\ qis}} \big(\operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(-,F(A'))\big) \tag{3}$$

from $\mathbf{D}(\mathcal{B})^{\mathrm{op}}$ to <u>Sets</u>. Then X is unique up to canonical isomorphism, which justifies writing $RF(A) = X$.

Let us make this explicit. For each $A \xrightarrow{q} A'$ in $qis\backslash A$, we get an arrow

$$\eta_{A'}\colon F(A') \to X$$

in $\mathbf{D}(\mathcal{B})$, corresponding via (3) to the the element represented by $1_{F(A')}$ on the right-hand side. The system of arrows $\eta_{A'}$ is compatible with $F(qis\backslash A)$, *i.e.*, for each $A' \xrightarrow{r} A''$ in $qis\backslash A$, we have $\eta_{A'} = \eta_{A''} \circ F(r)$. By naturality, the arrows $\eta_{A'}$ induce the map $\varinjlim_{A \xrightarrow[qis]{\cong} A'} \big(\operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(-,F(A'))\big) \to \operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(-,X)$.

In the opposite direction, to the element 1_X on the left-hand side of (3) there corresponds an equivalence class of arrows $\sigma_{A'}\colon X \to F(A')$, for some $A' \in qis\backslash A$. Any two representatives $\sigma_{A'}$ factor into a third, and by naturality, the map $\operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(-,X) \to \varinjlim_{A \xrightarrow[qis]{\cong} A'} \big(\operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(-,F(A'))\big)$ is induced by any representative $\sigma_{A'}$.

The two maps being inverse means that (i) any representative $\sigma_{A'}$ is a section of $\eta_{A'}$, (ii) the maps $\eta_{A'}\colon F(A') \to X$ make X the inductive limit $X = \varinjlim F(qis\backslash A)$ in $\mathbf{D}(\mathcal{B})$, and (iii) for any representative $\sigma_{A'}$, the arrow $X \to Y$ corresponding to any system of maps $\gamma_{A'}\colon F(A') \to Y$, $A' \in qis\backslash A$ by the universal property of $\varinjlim F(qis\backslash A)$ is given by $\gamma_{A'} \circ \sigma_{A'}$. In this case, the system $F(qis\backslash A)$ is said to be *essentially constant.*

The practical meaning of 4.6 will become clearer in §5, where we will give criteria that one uses in practice to show that $RF(A)$ is defined. The criteria also have the effect of making $RF(A)$ concretely computable, often in more than one way. But first we need to remain a little longer in the abstract context in order to establish the basic properties of RF.

Definition 4.8. The cohomology objects $H^i(RF)$ are denoted R^iF and called the *classical right derived functors of F.*

Corollary 4.9. *Let $\mathbf{D}^F(\mathcal{A})$ be the full subcategory of $\mathbf{D}(\mathcal{A})$ whose objects are those A such that $RF(A)$ is defined. Then RF is a functor from $\mathbf{D}^F(\mathcal{A})$ to $\mathbf{D}(\mathcal{B})$.*

Corollary 4.10. *The subcategory* $\mathbf{D}^F(\mathcal{A})$ *is closed under isomorphisms in* $\mathbf{D}(\mathcal{A})$.

Corollary 4.11. *If* $RF(A)$ *is defined then* $RF(A[n])$ *is defined and equal to* $RF(A)[n]$.

Corollary 4.12. *Suppose* F *maps quasi-isomorphisms to quasi-isomorphisms (in particular, if* F *comes from an exact functor* $F\colon \mathcal{A} \to \mathcal{B}$*). Then* RF *is defined on all of* $\mathbf{D}(\mathcal{A})$ *and is just the functor from* $\mathbf{D}(\mathcal{A})$ *to* $\mathbf{D}(\mathcal{B})$ *induced by* F*, via the universal property of* $\mathbf{D}(\mathcal{A})$.

Proof. In this case, $F(qis\backslash A)$ is a constant inductive system in $\mathbf{D}(\mathcal{B})$ with limit $F(A)$. ☐

Remark 4.13. Originally, Verdier defined a right derived functor of F (assuming one exists) to be a functor $RF\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$, together with a natural transformation $j_{\mathcal{B}} \circ F \to RF \circ j_{\mathcal{A}}$, satisfying the universal property that for any other such functor $G\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$, the transformation $j_{\mathcal{B}} \circ F \to G \circ j_{\mathcal{A}}$ factors through $\theta \circ j_{\mathcal{A}}$ for a unique natural transformation $\theta\colon RF \to G$. Verdier's definition is still the one found most often in the literature.

Historically, it was not always clear how to construct some important derived functors on all of $\mathbf{D}(\mathcal{A})$, so Verdier also allowed derived functors on a subcategory $\mathcal{D} \subseteq \mathbf{D}(\mathcal{A})$ (for instance, on $\mathbf{D}^+(\mathcal{A})$—see 5.4), defined by the same universal property, restricted to functors from $\mathcal{D}$ to $\mathbf{D}(\mathcal{B})$.

By construction, Deligne's RF has Verdier's universal property, but now among functors $\mathbf{D}(\mathcal{A}) \to \operatorname{Ind}(\mathbf{D}(\mathcal{B}))$. When Deligne's RF is defined on $\mathcal{D} \subseteq \mathbf{D}(\mathcal{A})$, it is then immediate that it is a right derived functor of F in the sense of Verdier.

It is not clear that existence of a Verdier derived functor, say $\mathcal{R}F$, must imply that RF is defined everywhere (whence $RF = \mathcal{R}F$). As it turns out, this question is unimportant, because in practice, the techniques one uses to construct a Verdier derived functor on $\mathcal{D}$ actually show that Deligne's RF is defined on $\mathcal{D}$.

Theorem 4.14 (Deligne [5])**.** *Let* $F\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ *be exact (e.g., if* F *comes from an additive functor* $F\colon \mathcal{A} \to \mathcal{B}$*). Let* $A \to B \to C \to A[1]$ *be an exact triangle in* $\mathbf{D}(\mathcal{A})$*. If* $RF(A)$ *and* $RF(B)$ *are defined, then* $RF(C)$ *is defined, and* $RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$ *is an exact triangle in* $\mathbf{D}(\mathcal{B})$.

We need the following lemma for the proof.

Lemma 4.15. *Let* $A \to B \to C \to A[1]$ *be an exact triangle in* $\mathbf{K}(\mathcal{A})$*. There exist morphisms of exact triangles in* $\mathbf{K}(\mathcal{A})$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "q"'] & B \arrow[r] \arrow[d, "r"'] & C \arrow[r] \arrow[d, "s"'] & A[1] \arrow[d, "{q[1]}"'] \\
A' \arrow[r, "u"] & B' \arrow[r, "v"] & C' \arrow[r, "w"] & A'[1]
\end{tikzcd} \tag{4}$$

with all vertical arrows quasi-isomorphisms, such that A', B', C' *are cofinal in* $qis\backslash A$, $qis\backslash B$, $qis\backslash C$*, respectively.*

Proof. Given (4) and $B' \xrightarrow{r'} B''$ in $qis\backslash B$, there is an exact triangle $A' \to B'' \to C'' \to A'[1]$ on $r'u\colon A' \to B''$, and, by 2.3(iii), a morphism from $A' \to B' \to C' \to A'[1]$ to $A' \to B'' \to$

$C'' \to A'[1]$, whose component arrows are all quasi-isomorphisms, by 3.12. This shows that the vertices B' in (4) are cofinal in $qis\backslash B$. The corresponding statement holds for the other vertices by 2.3(ii). $\square$

Proof of Theorem 4.14. Let $RF(A) = X$, $RF(B) = Y$. Complete the arrow $X \to Y$ to an exact triangle $X \to Y \to Z \to X[1]$ in $\mathbf{D}(\mathcal{B})$. As in 4.7, we can find sections $\sigma_{A'} \colon X \to F(A')$, $\sigma_{B'} \colon Y \to F(B')$. Changing A', B' if needed, we can fit these into a commutative diagram in $\mathbf{D}(\mathcal{B})$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\sigma_{A'}"'] & Y \arrow[d, "\sigma_{B'}"'] \\
F(A') \arrow[r, "F(u)"] & F(B')
\end{tikzcd},$$

and we can further assume that $u \colon A' \to B'$ is part of an exact triangle in the bottom row of (4). Since F is exact, this extends to a morphism of exact triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\
F(A') \arrow[r] & F(B') \arrow[r] & F(C') \arrow[r] & F(A')[1]
\end{tikzcd}. \tag{5}$$

Now, 4.15 and 3.24 imply that the sequence

$$\cdots \to \varinjlim_{A'} \operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(T, F(A')) \to \varinjlim_{B'} \operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(T, F(B')) \to \varinjlim_{C'} \operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(T, F(C')) \to \varinjlim_{A'} \operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(T, F(A')[1]) \to \cdots \tag{6}$$

is exact for every object T of $\mathbf{D}(\mathcal{B})$. By definition, this is just the sequence

$$\begin{tikzcd}
\cdots \arrow[r] & RF(A) \arrow[r] \arrow[d, equal] & RF(B) \arrow[r] \arrow[d, equal] & RF(C) \arrow[r] & RF(A)[1] \arrow[r] \arrow[d, equal] & \cdots \\
& \operatorname{Hom}(-, X) & \operatorname{Hom}(-, Y) & & \operatorname{Hom}(-X[1]) &
\end{tikzcd}$$

in $\operatorname{Ind}(\mathbf{D}(\mathcal{B}))$, evaluated at T. The morphism in (5) provides a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Hom}(-, X) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(-, Y) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(-, Z) \arrow[r] \arrow[d] & \operatorname{Hom}(-, X[1]) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & RF(A) \arrow[r] & RF(B) \arrow[r] & RF(C) \arrow[r] & RF(A)[1] \arrow[r] & \cdots
\end{tikzcd}.$$

Evaluated at any T in $\mathbf{D}(\mathcal{B})$, we have just seen that the bottom row is exact, and the top row is exact by 3.24. Hence the vertical arrow is an isomorphism. $\square$

Corollary 4.16. *For any triangle $A \to B \to C \to A[1]$ in $\mathbf{D}(\mathcal{A})$, and in particular, for any exact sequence $0 \to A \to B \to C \to 0$ in $\mathbf{C}(\mathcal{A})$, if $RF(A)$, $RF(B)$, $RF(C)$ are defined (e.g., if any two of them are), there is a long exact sequence of classical derived functors*

$$\cdots \to R^0F(A) \to R^0F(B) \to R^0F(C) \to R^1F(A) \to R^1F(B) \to \cdots .$$

5. Computing derived functors

Definition 5.1. Let $F\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ be an exact functor. A complex I in $\mathbf{K}(\mathcal{A})$ *computes*[1] RF if the canonical natural morphism $\eta_I\colon F(I) \to RF(I)$ in 4.7 is an isomorphism in $\mathbf{D}(\mathcal{B})$.

A quasi-isomorphism $A \xrightarrow{q} I$ is called a *resolution of* A. If I computes RF, then $RF(A)$ exists and is identified with $F(I)$, the natural morphism $F(A) \to RF(A)$ being given by $F(q)$. We will give some criteria for the existence of a resolution of A that computes RF.

Proposition 5.2. *If F sends quasi-isomorphisms $A \to A'$ to quasi-isomorphisms, then A computes RF.*

Proof. In this case, $F(qis\backslash A)$ is a constant inductive system (all its maps are isomorphisms in $\mathbf{D}(\mathcal{B})$), with limit $F(A)$. ☐

In particular, this gives another way to see 4.12.

Lemma 5.3. *Let $\mathfrak{I} \subseteq qis\backslash A$ be a class of complexes such that*

(i) For every resolution $A \underset{qis}{\xrightarrow{\simeq}} A'$ there is a resolution $A' \underset{qis}{\xrightarrow{\simeq}} I$ with $I \in \mathfrak{I}$, i.e., $\mathfrak{I}$ is cofinal in $qis\backslash A$, and in particular, A has a resolution $A \underset{qis}{\xrightarrow{\simeq}} I$ with $I \in \mathfrak{I}$;

(ii) For every quasi-isomorphism $q\colon I \underset{qis}{\xrightarrow{\simeq}} J$ with $I, J \in \mathfrak{I}$, $F(q)$ is a quasi-isomorphism.

Then every $I \in \mathfrak{I}$ computes RF. In particular, a resolution $A \underset{qis}{\xrightarrow{\simeq}} I$ induces $F(A) \xrightarrow{F(q)} RF(A) = F(I)$.

Proof. By hypothesis $F(\mathfrak{I})$ is cofinal in $F(qis\backslash A)$ and constant with limit $F(I)$. ☐

Definition 5.4. The *bounded-below* derived category $\mathbf{D}^+(\mathcal{A})$ is the full subcategory of $\mathbf{D}(\mathcal{A})$ consisting of objects A such that for some n_0, we have $H^i(A) = 0$ for all $i < n_0$. The *bounded-above* derived category $\mathbf{D}^-(\mathcal{A})$ is defined dually. The *bounded* derived category is $\mathbf{D}^b(\mathcal{A}) = \mathbf{D}^+(\mathcal{A}) \cap \mathbf{D}^-(\mathcal{A})$.

Remark 5.5. The *truncation functor* $\tau_{\geq n}$ sends a complex A to the complex

$$\tau_{\geq n}(A) = \cdots \to 0 \to (A^n/\operatorname{im}(d^{n-1})) \to A^{n+1} \to A^{n+2} \to \cdots .$$

Then

$$H^i(\tau_{\geq n}(A)) = \begin{cases} H^i(A) & \text{if } i \geq n \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\tau_{\geq n}$ preserves quasi-isomorphisms and hence is well-defined as an endo-functor on any of the categories $\mathbf{C}(\mathcal{A})$, $\mathbf{K}(\mathcal{A})$, $\mathbf{D}(\mathcal{A})$. There is an obvious canonical functorial surjection $A \to \tau_{\geq n}(A)$. The exact sequence $0 \to \tau_{<n}(A) \to A \to \tau_{\geq n}(A) \to 0$ defines the

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[1] In Deligne [5], I is *déployé pour F*, translated by Spaltenstein [8] as *unfolded for F*, but to my ear that doesn't have the right ring in English.

dual truncation functor $\tau_{<n}$ which kills the cohomology $H^{\geq n}(A)$. In $\mathbf{D}(\mathcal{A})$, this becomes a triangle

$$\tau_{<n}(A) \to A \to \tau_{\geq n}(A) \to \tau_{<n}(A)[1] = \tau_{<(n-1)}(A[1]).$$

If A is already bounded below at n_0, then $A \to \tau_{\geq n_0}(A)$ is a quasi-isomorphism. It follows immediately that $\mathbf{D}^+(\mathcal{A})$ is equivalent to its full subcategory of *strictly bounded-below* objects A, satisfying $A^i = 0$ for all i less than some n_0. Moreover, if $A \to X \underset{qis}{\overset{\simeq}{\leftarrow}} B$ is a morphism between objects $A, B \in \mathbf{D}^+(\mathcal{A})$, then necessarily $X \in \mathbf{D}^+(\mathcal{A})$. Truncating all three objects, we can replace X by a strictly bounded-below complex too. Hence $\mathbf{D}^+(\mathcal{A})$ can be identified with the localization $\mathbf{C}^+(\mathcal{A})[Q^{-1}]$ of the category $\mathbf{C}^+(\mathcal{A})$ of strictly bounded-below complexes by the quasi-isomorphisms in $\mathbf{C}^+(\mathcal{A})$.

Proposition 5.6. *Assume $F \colon \mathcal{A} \to \mathcal{B}$ is left exact. Let $\mathfrak{A}$ be a class of objects in $\mathcal{A}$ such that*

(i) For every A in $\mathcal{A}$ there is an injection $A \to I$ with $I \in \mathfrak{A}$;

(ii) $\mathfrak{A}$ is closed under finite direct sums, and if $0 \to I \to J \to N \to 0$ is an exact sequence with $I, J \in \mathfrak{A}$, then $N \in \mathfrak{A}$;

(iii) If $0 \to I \to A \to B \to 0$ is an exact sequence with $I \in \mathfrak{A}$, then $0 \to F(I) \to F(A) \to F(B) \to 0$ is exact.

Then every A in $\mathbf{D}^+(\mathcal{A})$ has a resolution $A \to I^\bullet$, where $I^\bullet$ is a strictly bounded-below complex of objects in $\mathfrak{A}$, and any such $I^\bullet$ computes RF, i.e., the resolution induces $F(A) \to RF(A) = F(I^\bullet)$.

Proof. We can assume that A is strictly bounded-below, say $A^i = 0$ for $i < 0$. Suppose by induction on k that we have constructed a homomorphism $r \colon A \to I^{(k)}$ as follows:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A^k \arrow[r] \arrow[d] & A^{k+1} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \arrow[r] & I^k \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd},$$

where the bottom row is a complex of objects in $\mathfrak{A}$, and r induces an isomorphism $H^i(A) \to H^i(I)$ for $i < k$ and an injection $H^k(A) \to \operatorname{coker}(d_I^{k-1})$ (initially, we have this with $k = -1$).

By 3.11, the mapping cone $C(r)$ has $H^i(C(r)) = 0$ for $i < k$. Then $C(r) \to J = \tau_{\geq k}(C(r))$ is a quasi-isomorphism, so $A \to I^{(k)} \to J \to A[1]$ is an exact triangle. Now J is strictly bounded below at k, hence the mapping cone $B = C(I^{(k)} \to J)[-1]$ looks like

$$0 \to I^0 \to I^1 \to \cdots \to I^k \to J^k \to J^{k+1} \to \cdots,$$

and there is a quasi-isomorphism $A \to B$, *i.e.*, B is a resolution of A. Choose an injection $i \colon J^k \hookrightarrow I^{k+1}$ with $I^{k+1} \in \mathfrak{A}$. Consider the homomorphism $B \to I^{(k+1)}$ given by

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "1"'] & I^0 \arrow[r] \arrow[d, "1"'] & I^1 \arrow[r] \arrow[d, "1"'] & \cdots \arrow[r] & I^k \arrow[r, "d^k"] \arrow[d, "1"'] & J^k \arrow[r] \arrow[d, "i"'] & J^{k+1} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \arrow[r] & I^k \arrow[r, "d'^k = id^k"] & I^{k+1} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}, \tag{7}$$

where $I^{(k+1)}$ is the bottom row. Composing with the quasi-isomorphism $A \to B$ we get a homomorphism $A \to I^{(k+1)}$. In (7), d^k and d'^k have the same kernel, so $H^k(A) = H^k(B) \to H^k(I^{(k+1)})$ is an isomorphism. Similarly, d^k and d'^k have the same image, and since i is injective, this implies that $H^{k+1}(A) = H^{k+1}(B) \hookrightarrow \operatorname{coker}(d'^k)$. So $I^{(k+1)}$ again satisfies the induction hypothesis. The limit $I^\bullet = I^0 \to I^1 \to \cdots$ is the desired resolution of A.

Let $I \xrightarrow{q} J$ be a quasi-isomorphism between bounded-below complexes of objects in $\mathfrak{A}$. Then $C(q)$ is a bounded-below acyclic complex of objects in $\mathfrak{A}$, and from (ii, iii) it follows easily by induction on the cohomology degree that $F(C(q)) = C(F(q))$ is acyclic. Hence $F(q)$ is a quasi-isomorphism. By the first part of the proof, every resolution of I has a strictly bounded-below resolution J by a complex of objects in $\mathfrak{A}$. Then 5.3 shows that I computes RF. □

Objects A in a class $\mathfrak{A}$ satisfying 5.6(i–iii) are said to be *acyclic* for F. It follows from 5.6 that they satisfy $R^iF(A) = 0$ for all $i > 0$. Conversely, using 4.16, one sees that the class $\mathfrak{A}$ of *all* objects A such that $R^iF(A) = 0$ for all $i > 0$ satisfies (i–iii). The value of 5.6 is that it enables us to recognize a class of acyclic objects without knowing how to calculate RF in advance.

Example 5.7. Let $\mathcal{A}$ be the category of (left) R-modules for any (possibly non-commutative) ring R, or the category of sheaves of $\mathcal{O}_X$-modules, where X is a ringed space. Then every object A of $\mathcal{A}$ has an injection $A \to I$ into an injective object. Any exact sequence as in 5.6(iii) with I injective splits, and injective objects satisfy 5.6(ii). Hence the injectives satisfy (i–iii) for any left-exact functor F. We conclude that RF is always defined on $\mathbf{D}^+(\mathcal{A})$, and that $RF(A) = F(I)$, where I is a strictly bounded-below injective resolution of A.

Remark 5.8. Spaltenstein [8] defines a complex I to be *K-injective* if the functor $\operatorname{Hom}^\bullet(-, I)$ (1.8) from $\mathbf{C}(\mathcal{A})$ to itself is exact. If I consists of a single object, this is equivalent to I being injective. Spaltenstein shows that certain special inverse limits of K-injective complexes are again K-injective. In particular, this implies that any bounded-below complex of injectives is K-injective. But more is true: if $\mathcal{A}$ is the category of R-modules or of $\mathcal{O}_X$-modules, Spaltenstein constructs a K-injective resolution of an arbitrary (unbounded) complex.

K-injectivity of I is equivalent to the property that $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(-, I) = \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(I)$ (exercise, using 1.8). In particular, if $I \xrightarrow{q} J$ is a quasi-isomorphism of K-injective complexes, then q is a homotopy equivalence. For any functor $F\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$, it follows that $F(q)$ is again a homotopy equivalence and hence a quasi-isomorphism. Using 5.3, it follows that RF is defined on all of $\mathbf{D}(\mathcal{A})$ for any F, and $RF(A) = F(I)$ for any K-injective resolution $A \to I$.

Example 5.9. Let $f\colon X \to Y$ be a morphism of ringed spaces. A sheaf $\mathcal{F}$ on X is called *flasque* if the restriction map $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is surjective, for every open $U \subseteq X$. It is not hard to show that flasque sheaves satisfy 5.6(ii, iii) for the direct image functor f_*. Moreover, injective sheaves are flasque, hence flasque sheaves satisfy (i). It follows that flasque sheaves are acyclic for f_*, and for any bounded-below complex of sheaves A on X, we have $Rf_*(A) = f_*(J)$, where J is a strictly bounded-below flasque resolution.

Now, it is immediate from the definition that if $\mathcal{F}$ is flasque, then so is $f_*(\mathcal{F})$ (this property does not hold for injective sheaves). Let $A \to J$ be a flasque resolution. Then we have

$$Rg_* Rf_*(A) = Rg_*(f_*(J)) = g_* f_*(J) = (gf)_*(J) = R(gf)_*(A),$$

i.e., there is a natural isomorphism of functors $Rg_* \circ Rf_* \cong R(gf)_*$ defined on $\mathbf{D}^+(\mathcal{A})$. The counterpart in terms of classical derived functors of this simple result is a spectral sequence relating the functors $R^p f_* \circ R^q g_*$ to $R^n(gf)_*$. The old style spectral sequence formula is not only much more complicated than the identity $Rg_* \circ Rf_* \cong R(gf)_*$, but it is also a weaker result.

Spaltenstein [8] gave a definition of *K-flasque complex* of sheaves. Using this, he proved that the identity $Rg_* \circ Rf_* \cong R(gf)_*$ also holds for unbounded complexes.