Homework problems for Lecture 9

1. Derive a formula for
(a) the number of (unlabelled) ordered rooted forests of k trees on n vertices;
(b) the number of labelled rooted forests of k trees on n vertices.

2. Prove that the number of ordered rooted trees with $n+1$ vertices and j leaves is equal to

$$\frac{1}{n+1}\binom{n+1}{j}\binom{n-1}{n-j}.$$

3. Prove that the number of ways to subdivide an n-gon into k polygons by introducing $k-1$ diagonals that do not intersect except at their endpoints is equal to the number of ordered rooted trees with $n+k-1$ vertices, $n-1$ leaves, and no vertices with exactly one child. Derive the formula

$$\frac{1}{n+k-1}\binom{n+k-1}{n-1}\binom{n-3}{n-k-2}$$

for this number. In particular, taking $k=n-2$, deduce that the number of triangulations of an n-gon is the Catalan number C_{n-2}. In this problem the n-gon is regarded as fixed in place, so for example the two triangulations of a square count as different even though they are the same up to symmetry. See Stanley, Exercise 6.19 for 65 more combinatorial interpretations of Catalan numbers.