Math 249 Algebraic Combinatorics Spring, 2004

Homework problems for Lecture 7

1. Find the exponential generating function $D(x) = \sum_n D_n x^n/n!$, where D_n is the number of permutations $\sigma \in S_n$ with no fixed points. Deduce an explicit formula for D_n. [Permutations without fixed points are also called *derangements*. Compare Stanley, Example 2.2.1, where they are counted using the principle of inclusion and exclusion.]

2. Find the exponential generating function counting *involutions*, that is, permutations $\sigma \in S_n$ such that $\sigma^2 = 1$.

3. Let $g(n, e)$ denote the number of connected simple graphs (*i.e.*, without loops or multiple edges) on n labelled vertices with e edges. Derive the mixed ordinary/exponential generating function

$$\sum_{n=1}^{\infty} \sum_{e} g(n, e) q^e x^n/n! = \log \sum_{n=0}^{\infty} (1 + q)^{\binom{n}{2}} x^n/n!$$

and use it to compute $\sum_e g(n, e)q^e$ for all $n \leq 4$. As a check, count the graphs in question by hand and compare answers.

4. A rooted tree is *strictly binary* if every node is either a leaf (*i.e.*, has no children) or has exactly two children. It is a strictly binary plane tree if, in addition, there is an ordering specified on the two children of each non-leaf node.

(a) Show that every strictly binary tree with n leaves has $2n - 1$ nodes.

(b) Find exponential generating functions counting strictly binary trees and strictly binary plane trees on n (labelled) nodes.

(c) Show that the number of strictly binary plane trees with $n + 1$ leaves is the Catalan number C_n.

5. A *leaf-labelled* tree T is a rooted trees whose leaves are the elements of a given set S, and whose other nodes are unlabelled. Let us require each non-leaf node of T to have at least two children; then there are finitely many leaf-labelled trees on a given finite leaf set S.

Introduce indeterminates $a_2, a_3, \ldots$ and assign each leaf-labelled tree the weight $w(T) = \prod_i a_i^{r_i}$, where r_i is the number of nodes in T with i children. Then show that the mixed ordinary/exponential generating function enumerating leaf-labelled trees with these weights is the functional composition inverse $F(x) = (x - A(x))^{\langle -1 \rangle}$, where $A(x) = \sum_{n\geq 2} a_n x^n/n!$.