Homework problems for Lecture 5

1. The following two identities are due to Gauss:

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} = \prod_{i\geq 1}\frac{1-q^i}{1+q^i};$$
$$\sum_{n\geq 0} q^{\binom{n+1}{2}} = \prod_{i\geq 1}\frac{1-q^{2i}}{1-q^{2i-1}}.$$

(a) Interpret them combinatorially as partition identities.

(b) Prove them, either combinatorially (not so easy) or using Jacobi's triple product identity (see problems for Lecture 4).

2. Construct an involution σ on the set of all partitions with distinct parts, with the following properties.

(i) For all λ, $|\sigma(\lambda)| = |\lambda|$.

(ii) λ is a fixed point of σ if and only if $\lambda = \emptyset$ or it has one of the forms $\lambda = (2k-1, 2k-2, \ldots, k)$ or $\lambda = (2k, 2k-1, \ldots, k+1)$.

(iii) For any other λ, one obtains $\sigma(\lambda)$ either by deleting the smallest part of λ, say $\lambda_l = k$, and adding 1 to the k largest parts of what remains, or by subtracting 1 from each of the k largest parts of λ, and adding a new smallest part equal to k to what remains.

(iv) The parts to which ± 1 is added in (iii) are all consecutive.

From the existence of σ satisfying (i) and (ii) and changing the number of parts of every non-fixed point λ by ± 1 as in (iii), deduce Euler's pentagonal number theorem

$$\prod_{i=1}^{\infty}(1-q^i) = 1 + \sum_{k\geq 1}(-1)^k\left(q^{(3k^2-k)/2} + q^{(3k^2+k)/2}\right).$$

This proof is originally due to F. Franklin (Comptes Rendus **82**, 1881).