Math 249 Algebraic Combinatorics Spring, 2004

Homework problems for Lecture 3

1. Prove *Cauchy's identity:*

$$\prod_{i\geq 0} \frac{1-axq^i}{1-xq^i} = \sum_{n\geq 0} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} x^n,$$

by showing that it reduces to the $q$-binomial theorem upon setting $a = q^m$ for an integer $m$ (note that you get either form of the $q$-binomial theorem by taking $m$ positive or negative).

2. (a) Give a direct combinatorial proof of the partition identities

$$\prod_{i\geq 1} \frac{1}{1-xq^i} = \sum_{n\geq 0} \frac{x^n q^n}{(1-q)(1-q^2)\cdots(1-q^n)}$$

and

$$\prod_{i\geq 1}(1+xq^i) = \sum_{n\geq 0} \frac{x^n q^{\binom{n+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

(b) Show that the two identities are special cases of Cauchy's identity, above, and that they are limiting cases of the $q$-binomial theorem.

3. (a) Prove that the number of partitions of $n$ with no parts divisible by $d$ is equal to the number of partitions of $n$ with no part repeated $d$ or more times, for all $n$ and $d$.

(b) Prove that the number of partitions of $n$ in which each part $j$ is repeated less than $j$ times is equal to the number of partitions of $n$ in which no part is a square.

4. Let $p_+(n)$ be the the number of partitions of $n$ with an even number of parts and $p_-(n)$ the number with an odd number of parts. Let $p_{DO}(n)$ be the number of partitions of $n$ with distinct odd parts, and let $k(n)$ be the number of partitions of $n$ which are conjugate to themselves. Prove that

$$k(n) = p_{DO}(n) = (-1)^n(p_+(n) - p_-(n))$$