Homework problems for Lecture 27

1. The meet and join operations in a lattice are commutative, associative and idempotent ($x \wedge x = x$, $x \vee x = x$), and satisfy the *absorption laws*

$$x \wedge (x \vee y) = x = x \vee (x \wedge y).$$

Prove that conversely, if $(L, \wedge, \vee)$ is any set with two operations satisfying the above identities, then L is a lattice.

2. A *projective plane* consists of a set X of points and a collection of subsets of X, called lines, such that

(i) Every two distinct points are contained in a unique line,

(ii) The intersection of two distinct lines is a unique point.

The projective plane is *nondegenerate* if it contains four points, no three of which lie on a line.

Show that if L is a complemented modular lattice of length 3, not of the form $\mathbf{2} \times L'$, then its atoms and coatoms form the points and lines of a nondegenerate projective plane (identifying each coatom with the set of atoms below it), and conversely. In particular, taking the lattice $L_3(q)$, there is a projective plane with $q^2 + q + 1$ points for each prime power q. It is not hard to prove that a nondegenerate finite projective plane always has $q^2 + q + 1$ points for some integer q, called its *order*. It is known that there exist nondegenerate finite projective planes with $L \not\cong L_n(q)$, but it is a famous unsolved problem whether q must always be a power of a prime.

3. A subset X of the atoms of a geometric lattice L is called *independent* if no element of X is under the join of the others. Show that the following are equivalent:

(i) X is independent;

(ii) $r(\bigvee X) = |X|$;

(iii) The join-sublattice generated by X is a Boolean algebra.

Also show that every element of L is the join of some independent set of atoms, and that every maximal independent set has join equal to $\mathbf{1}$. Deduce that a geometric lattice is always complemented.

4. (a) Given a finite lattice L, define $I(L)$ to be the poset of all intervals $[x, y] \subseteq L$, plus the empty set, ordered by containment. Show that $I(L)$ is a lattice and describe its meet and join.

(b) Show that the Möbius function in $I(L)$ is given by

$$\begin{aligned} &\mu([x, y], [w, z]) = \mu_L(w, x)\mu_L(y, z) \quad \text{for nonempty } [x, y], \\ &\mu(\emptyset, [w, z]) = -\mu_L(w, z). \end{aligned}$$

(c) Prove *Crapo's lemma*: let X be a subset of L, and let n_k be the number of k-element subsets of X with join equal to $\mathbf{1}$ and meet equal to $\mathbf{0}$. Then

$$\sum_k (-1)^k n_k = -\mu(\mathbf{0}, \mathbf{1}) + \sum_{\substack{x \leq y \\ [x,y] \cap X = \emptyset}} \mu(\mathbf{0}, x)\mu(y, \mathbf{1}).$$

(d) Stanley, Chapter 3, Exercise 33. The identity there is called *Crapo's complementation theorem*. This gives a second proof that geometric lattices are always complemented.