

## Homework problems for Lecture 26

1. (a) Show that if finite posets  $P$  and  $Q$  have isomorphic incidence algebras (over  $\mathbb{Q}$ ), then  $P \cong Q$  (Stanley, Ch. 3, Ex. 29).

(b) Let  $I(P/\cong)$  denote the reduced incidence algebra of functions on  $\text{Int}(P)$  constant on isomorphism classes of intervals. Find non-isomorphic posets  $P$  and  $Q$  such that  $I(P/\cong)$  and  $I(Q/\cong)$  are isomorphic.

2. The faces of an  $n$ -dimensional hypercube, including the cube itself and the empty face, form a poset  $Q_n \cong \mathbf{0} \oplus \wedge^n$ , where  $\wedge$  is the 3-element poset with two minimal elements and one maximal element (see Stanley, Ch. 3, Ex. 31 for more information).

(a) Show that every interval in  $Q_n$  is isomorphic to some  $Q_k$  or to a Boolean algebra  $B_k$ .

(b) Let  $\mathcal{Q}$  be the family of all intervals in all  $Q_n$ . Let  $\sim$  denote all isomorphisms, except we do not include the isomorphisms  $Q_0 \cong B_1$  or  $Q_1 \cong B_2$ , and we consider intervals  $[x, y] \subseteq Q_n$  of length 1 or 2 as isomorphic to  $Q_0$  or  $Q_1$  if  $x = \mathbf{0}$ , or to  $B_1$  or  $B_2$  otherwise. Show that the reduced incidence algebra

$$I(\mathcal{Q}/\sim)$$

(over  $\mathbb{Q}$ ) is isomorphic to the algebra of pairs  $(F_0(x), F_1(x))$  of formal power series, with the product defined by

$$(F_0(x), F_1(x)) \cdot (G_0(x), G_1(x)) = (F_0(x)G_0(x), F_0(0)G_1(x) + F_1(x)G_0(x)).$$

Identify the pairs  $(F_0, F_1)$  corresponding to the identity element, the zeta function, and the Möbius function.

3. Let  $\mathcal{P}$  consist of all finite posets isomorphic to a product of chains. Let  $C_k = [k + 1]$  be the chain of length  $k$ , and for every partition  $\lambda$ , put  $P_\lambda = C_{\lambda_1} \times \cdots \times C_{\lambda_\ell}$ .

Let  $\widehat{\Lambda}$  be the algebra of symmetric formal series, and define a map

$$\phi: I(\mathcal{P}, \cong) \rightarrow \widehat{\Lambda}$$

by  $\phi f = \sum_\lambda f(P_\lambda) m_\lambda$ . Show that  $\phi$  is an algebra isomorphism, and identify  $\phi(\zeta)$  and  $\phi(\mu)$ . Which elements of  $I(\mathcal{P}, \cong)$  correspond to the elementary symmetric functions  $e_n$ , the complete homogeneous symmetric functions  $h_n$ , and the power-sums  $p_n$ ?