Homework problems for Lecture 25

1. Let χ be the character of an irreducible representation W of a finite group G. Prove that the element $\sum_{g\in G} \chi(g^{-1})g \in \mathbb{C}G$ generates a submodule $V \subseteq \mathbb{C}G$ isomorphic to the direct sum of $\dim(W)$ copies of W. In particular, V has a submodule of dimension $\dim(W)$ and any such submodule is isomorphic to W.

Note that this provides a general procedure for constructing irreducible representations, but it is unwieldy if the group is large (it's not practical if $G = S_{10}$, for example).

2. Describe explicitly a 5-dimensional irreducible representation $\rho\colon S_5 \to GL_5$, by giving matrices $\rho(\sigma)$ for some elements σ that generate S_5. [Labor-saving hint: S_n can be generated by two elements.]

3. Define the *Kronecker product* on symmetric functions in terms of the power-sum basis by

$$p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda.$$

Equivalently, the symmetric functions p_λ/z_λ are orthogonal idempotents with respect to $*$. Prove that the Kronecker coefficients $a^\lambda_{\mu\nu}$ defined by

$$s_\mu * s_\nu = \sum_\lambda a^\lambda_{\mu\nu} s_\lambda$$

are non-negative integers.

It is an open problem to find a combinatorial rule for the computation of Kronecker coefficients, except in some special cases.

4. Let W be a G-module. Choose a complete set of mutually non-isomorphic irreducible representations V_i of G and set $T_i = \mathrm{Hom}_G(V_i, W)$. Show that $W \cong \bigoplus_i V_i \otimes T_i$, where each vector space T_i is regarded as a G-module with trivial G-action.

5. Let V be an m-dimensional complex vector space, and let $GL(V)$ be the group of linear automorphisms of V. We can consider V as a $GL(V)$-module; choosing a basis of V we get the usual matrix representation of $GL(V)$ isomorphically onto $GL_m(\mathbb{C})$. More generally, a group homomorphism $\rho\colon GL(V) \to GL(W)$ is called a *polynomial representation* of $GL(V)$ if, after choosing bases in V and W, the matrix entries of $\rho(g)$ are polynomial functions of the matrix entries of g.

(a) Given a finite-dimensional polynomial representation (W, ρ) of $GL(V)$, show that there is a symmetric function χ_W such that $\mathrm{tr}(\rho(g), W) = \chi_W(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are the eigenvalues of $g \in GL(V)$. Using plethystic notation, this can also be written $\mathrm{tr}(\rho(g), W) = \chi_W[\mathrm{tr}(g)]$. The symmetric function χ_W is called the *character* of the representation (W, ρ).

(b) Show that the characters χ_ρ share properties of characters of finite groups: in particular, $\chi_{W_1\oplus W_2} = \chi_{W_1} + \chi_{W_2}$ and $\chi_{W_1\otimes W_2} = \chi_{W_1}\chi_{W_2}$.

(c) The n-th tensor power $V^{\otimes n}$ is an S_n-module, where S_n acts by permuting the tensor factors. Then $GL(V)$ acts on $V^{\otimes n}$ by S_n-module homomorphisms, inducing an action of $GL(V)$ on $S_\lambda(V) = \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n})$ for each irreducible representation V_λ of S_n. (S_λ is called a *Schur functor*). Show that $S_\lambda(V)$ is a polynomial representation of $GL(V)$ and that its character is given by the Schur function

$$\chi_{S_\lambda(V)} = s_\lambda(x).$$

In particular, $S_\lambda(V) = 0$ if $l(\lambda) > m = \dim(V)$. [Hint: Compute the character of $V^{\otimes n}$ as an $S_n \times GL(V)$-module, then decompose $V^{\otimes n}$ into submodules isomorphic to $V_\lambda \otimes S_\lambda(V)$—see Problem 4.]

(d) Show that the one-dimensional representation of $GL(V)$ given by $\rho(g) = \det(g)$ has character e_m, and more generally that the representation $\rho(g) = \det(g)^k$ has character $s_{(k^m)}$.

(e) Show that the exterior powers $\bigwedge^k(V)$ have character e_k, and the symmetric powers $S^k(V)$ have character h_k.

(f) $GL(V)$ is a *reductive algebraic group*, which means that every finite-dimensional polynomial representation is a direct sum of irreducible ones. It can be shown that the representations $S_\lambda(V)$ for $l(\lambda) \leq m$ form a complete set of mutually non-isomorphic polynomial representations of $GL(V)$. In particular, this implies that the character of every polynomial representation of $GL(V)$ is a linear combination of Schur functions with non-negative integer coefficients. Use this to prove that

$$s_\mu[s_\nu[X]] = \sum_\lambda d^\lambda_{\mu\nu} s_\lambda[X],$$

where the *plethysm coefficients* $d^\lambda_{\mu\nu}$ are non-negative integers. It is an open problem to find a combinatorial rule for the computation of plethysm coefficients, except in some special cases.