Homework problems for Lecture 18

1. Consider the partition (a^b) whose diagram is an $a \times b$ rectangle. Prove that

$$s^2_{(a^b)} = \sum_{\mu} s_\mu,$$

where all coefficients are equal to 1 and the sum ranges over partitions μ containing (a^b) and such that $\mu/(a^b)$ is the disjoint union of (translates of) partition diagrams $\rho, \nu \subseteq (a^b)$, satisfying $\nu = ((a^b)/\rho)^\perp$. Here $(-)^\perp$ denotes rotation of a diagram through 180°.

2. Let I be the ideal in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ generated by the elementary symmetric functions $e_1(x), \ldots, e_n(x)$.

(a) Prove that if $f_1, \ldots, f_n$ is any set of homogeneous generators of the ring of symmetric functions $\Lambda_{\mathbb{Z}}(x_1, \ldots, x_n)$, then $I = (f_1, \ldots, f_n)$; in particular $I = (h_1(x), \ldots, h_n(x))$.

(b) Prove that I is generated by the "partial" symmetric polynomials $h_k(x_k, \ldots, x_n)$, for $k = 1, \ldots, n$. Hint: $h_k(x_k, \ldots, x_n) = h_k[X - X_{k-1}]$, where $X_k = x_1 + \cdots + x_k$.

Note: this problem really goes with Lecture 14, but I didn't think to assign it then.

3. Let $Q_{n,D}(x)$ be the fundamental quasisymmetric function (denoted $L_{\mathrm{co}(D)}$ in Stanley's book) in infinitely many variables $x = x_1, x_2, \ldots$. Set $\mathcal{Q}^{(n)} = \mathbb{Z} \cdot \{Q_{n,D}(x) : D \subseteq [n-1]\}$ and $\mathcal{Q} = \bigoplus_n \mathcal{Q}^{(n)}$.

(a) Verify (cf. Stanley §7.19) that $\mathcal{Q}$ is a ring, containing the ring of symmetric functions Λ as a subring, and that the functions $Q_{n,D}(x)$ for all n and D form a $\mathbb{Z}$-basis of $\mathcal{Q}$.

(b) Define an involution $\omega\colon \mathcal{Q} \to \mathcal{Q}$ by $\omega(Q_{n,D}) = Q_{n,[n-1]\setminus D}$. Show that ω is a ring homomorphism, and that it maps Λ into itself and coincides on Λ with the standard involution ω.

4. Let $\mathcal{N}^{(n)} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{Q}^{(n)}, \mathbb{Z})$ be the dual lattice to $\mathcal{Q}^{(n)}$ and set $\mathcal{N} = \bigoplus_n \mathcal{N}^{(n)}$. Let $Q(n,D)(x,y)$ denote $Q_{n,D}$ evaluated on an ordered alphabet $\{x_1 < x_2 < \cdots < y_1 < y_2 < \cdots\}$. Identifying $\mathcal{Q} \otimes_{\mathbb{Z}} \mathcal{Q}$ with $\mathcal{Q}(x)\mathcal{Q}(y)$, define a map $\Delta\colon \mathcal{Q} \to \mathcal{Q} \otimes_{\mathbb{Z}} \mathcal{Q}$ by $\Delta(Q_{n,D}) = Q_{n,D}(x,y)$. Let $\mu\colon \mathcal{N} \otimes_{\mathbb{Z}} \mathcal{N} \to \mathcal{N}$ be the dual map.

(a) Show that $\mathcal{N}$ is an associative ring with multiplication defined by μ.

(b) Define $h_n \in \mathcal{N}^{(n)}$ by $h_n(Q_{n,D}) = \delta_{D,\emptyset}$, and e_n by $e_n(Q_{n,D}) = \delta_{D,[n-1]}$. Show that $\mathcal{N} = \mathbb{Z}\langle h_1, h_2, \ldots \rangle$ is the free associative (but not commutative) algebra generated by the h_n's, and that $\mathcal{N} = \mathbb{Z}\langle e_1, e_2, \ldots \rangle$ also.

(c) Show that the linear functionals e_n, h_n on $\mathcal{Q}$, when restricted to Λ, coincide with the Hall inner products $\langle e_n, - \rangle$ and $\langle h_n, - \rangle$ respectively.