Homework problems for Lecture 15

1. Let ϵ be a fictitious alphabet such that $p_k[\epsilon] = \delta_{1,k}$. Stated more correctly, this means we are to interpret $f[\epsilon]$ as the image of f under the homomorphism $\Lambda \to \mathbb{Q}$ mapping p_k to $\delta_{1,k}$.

(a) Prove the identity $f[\epsilon] = \langle f, \exp(p_1)\rangle$.

(b) Prove the identity $f[\epsilon] = \lim_{n\to\infty}(f[nx])_{x\mapsto 1/n}$.

(c) Show that $e_k[\epsilon] = h_k[\epsilon] = 1/n!$.

(d) More generally, show that $s_\lambda[\epsilon] = f_\lambda/n!$, where $|\lambda| = n$ and f_λ is the number of standard Young tableaux of shape λ. (A standard Young tableau is a semistandard tableaux whose entries are $1, 2, \ldots, n$.)

2. (a) Recall from class that $h_n = \sum_{|\lambda|=n} p_\lambda/z_\lambda$, where $z_\lambda = \prod_i i^{r_i} r_i!$ for $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$. Show that this is equivalent to Newton's determinant formula

$$h_n = \frac{1}{n!} \det \begin{bmatrix} p_1 & -1 & 0 & \ldots & 0 \\ p_2 & p_1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n-1} & p_{n-2} & \cdot & \ldots & -(n-1) \\ p_n & p_{n-1} & \cdot & \ldots & p_1 \end{bmatrix}$$

(b) Show that e_n is given by the same determinant without the minus signs.

3. Prove the identity $s_{(n-1,n-2,\ldots,1)}(x_1, \ldots, x_n) = \prod_{1\le i<j\le n}(x_i + x_j)$.

4. Let $|\lambda| = |\mu| = n$. Show that $\langle h_\lambda, h_\mu\rangle$ is equal to the number of double cosets $S_\lambda w S_\mu$ in the symmetric group S_n, where $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_l}$, embedded as a subgroup of S_n in the obvious way, and similarly for S_μ.

Note: Problems 2–4 above are from I. G. Macdonald: *Symmetric Functions and Hall Polynomials*.