Math 249 Algebraic Combinatorics Spring, 2004

Homework problems for Lecture 10

1. This problem suggests an alternate proof of the Matrix–Tree theorem, taken in the form

$$\det(M_n + \operatorname{diag}(z_1, \ldots, z_n)) = \sum_{\substack{\text{rooted forests } F \\ \text{on } \{1,\ldots,n\}}} \left(\prod_{i \in \operatorname{roots}(F)} z_i \right) \operatorname{wt}(F).$$

(a) Set $z_i = x_{ii}$ and show that every monomial appearing in $\det(M_n + \operatorname{diag}(x_{11}, \ldots, x_{nn}))$ has the form $m_f(x) = \prod_{i=1}^n x_{i,f(i)}$ for some function $f\colon [n] \to [n]$.

(b) Evaluate the coefficient of $m_f(x)$ by setting $x_{ij} = 1$ for $j = f(i)$, zero otherwise, in $\det(M_n + \operatorname{diag}(x_{11}, \ldots, x_{nn}))$. Show that the result is equal to zero if f has any cycle $v, f(v), f(f(v)), \ldots, f^{\langle k \rangle}(v) = v$ of length $k > 1$, and otherwise equal to one. Note that by symmetry among the vertex labels, you can assume each cycle of length greater than one has the form $i, f(i) = i+1, f(i+1) = i+2, \ldots, f(i+k-1) = i$, and that $f(j) > j$ for all other j. The matrix then takes a nice block-triangular form.

2. Let $f_m(r)$ be the number of rooted spanning forests with r roots in the graph C_m, a cycle on m vertices ($m > 1$). Prove that

$$F_m(z) \underset{\text{def}}{=} \sum_r f_m(r) z^r = \prod_{j=0}^{m-1} (z + 2 - 2\cos(2\pi j/m)) = \sum_{r=1}^{m} \frac{m}{r} \binom{m+r-1}{2r-1} z^r.$$

3. The product $G \times H$ of two simple graphs (graphs without loops or multiple edges) is the graph on vertex set $V(G) \times V(H)$ with edges $\{(v,w),(v',w')\}$ for $v = v'$ and $\{w,w'\} \in E(H)$ or $w = w'$ and $\{v,v'\} \in E(G)$. The adjacency matrix A_G of a graph G on n vertices is the $n \times n$ matrix with rows and columns labelled by the vertices, and entries $(A_G)_{v,w} = 1$ if $\{v,w\} \in E(G)$, zero otherwise. Let D_G be the diagonal matrix whose (v,v) entry is the degree of v.

(a) Let $f_G(r)$ be the number of rooted spanning forests of G with r roots, and let $F_G(z) = \sum_r f_G(r) z^r$ be the corresponding generating function. Show that $F_G(z) = \prod_i (z + \alpha_i)$, where the α_i's are the eigenvalues of $D_G - A_G$.

(b) Show that $F_{G \times H}(z) = \prod_{i,j} (z + \alpha_i + \beta_j)$, where $F_G(z) = \prod_i (z + \alpha_i)$ and $F_H(z) = \prod_j (z + \beta_j)$. In particular, the numbers $f_G(r)$ and $f_H(r)$ for all r determine the corresponding numbers $f_{G \times H}(r)$.

(c) Show that if Q_n is the graph formed by the vertices and edges of the n-cube, or equivalently, the Hasse diagram of the Boolean algebra of order n, then

$$F_{Q_n}(z) = \prod_{k=0}^{n} (z + 2k)^{\binom{n}{k}}.$$

This generalizes Stanley, Ex. 5.6.10, which follows by taking the coefficient of z.

4. Let G be an undirected simple graph with vertex set $[n]$ and all vertex degrees d_i even. Let M_G be the square matrix with off-diagonal entries

$$\begin{array}{cl} \dfrac{x_j}{x_i + x_j} & \text{if } \{i, j\} \text{ is an edge of } G, \\ 0 & \text{otherwise,} \end{array}$$

and all row-sums zero. Show that the number of closed Eulerian walks in G is given by the coefficient of $\lambda x_1^{d_1/2} \cdots x_n^{d_n/2}$ in

$$\frac{|E(G)|}{n} \prod_i (d_i/2 - 1)! \prod_{\{i,j\} \in E(G)} (x_i + x_j)\ \det(M_G + \lambda I).$$