1. (a) (10 pts) Find all values of x which are critical numbers for the function

$$f(x) = \arctan(2x) - |x+1|$$

$x = -1$ is a critical number because $f'(-1)$ D.N.E. For $x \neq -1$,

$$f'(x) = \begin{cases} \dfrac{2}{1+4x^2} - 1 & x > -1 \\ \dfrac{2}{1+4x^2} + 1 & x < -1 \end{cases}$$

The second case $(x<-1)$ has $f'(x)>0$. The first $(x>-1)$ has $f'(x)=0$ for $x = \pm 1/2$. So the critical numbers are $x=-1$, $x=-1/2$, $x=1/2$

(b) (10 pts) Find all local and absolute minima and maxima of $f(x)$. Give the value of x and $f(x)$ for each one, and say what kind it is.

You may find the table below of values of $\arctan(x)$ helpful.

$\arctan(-2) \approx -1.107$

$\arctan(-1) = -\pi/4 \approx -0.785$

$\arctan(0) = 0$

$\arctan(1) = \pi/4 \approx 0.785$

$\arctan(2) \approx 1.107$

$f(x)$ is continuous for all x. The sign of $f'(x)$ between critical numbers is

So $f(-1) = \arctan(-2)$ is a local max

$f(-\frac{1}{2}) = \arctan(-1) - \frac{1}{2} = -\frac{\pi}{4} - \frac{1}{2}$ is a local min

$f(\frac{1}{2}) = \arctan(1) - \frac{3}{2} = \frac{\pi}{4} - \frac{3}{2}$ is a local max.

There is no abs min since $f(x) \to -\infty$ as $x \to \pm\infty$.

Of the two local maxes, $f(-1) \approx -1.107$, $f(\frac{1}{2}) \approx -.715$, $f(\frac{1}{2})$ is larger and is the absolute max.

2. (10 pts) Find the limit

$$\lim_{x \to 0} \frac{\ln(\cos x)}{x^2}.$$

Use L'Hospital twice:

$$\lim_{x\to 0} \frac{\ln(\cos x)}{x^2} \underset{\text{"0/0"}}{=} \lim_{x\to 0} \frac{-\sin x/\cos x}{2x}$$

algebra

$$= \lim_{x\to 0} \frac{-\tan x}{2x}$$

$$\underset{\text{"0/0"}}{=} \lim_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}$$

direct sub.

3. (a) (7 pts) Show that the equation

$$4x^3 + 2 = 5x^2 + 2x$$

has a solution in the interval $[0, 1]$.

Let $f(x) = 4x^3 - 5x^2 - 2x + 2$ be the difference.
$f(0) = 2$, $f(1) = 6 - 7 = -1$, and $f(x)$ is continuous, so
$f(x) = 0$ has a solution in $[0,1]$ by intermediate value theorem.

(b) (8 pts) Suppose you use Newton's method to try to find the solution, taking $x_1 = 0$ as your first approximation. What do you get for the second and third approximations?

$f'(x) = 12x^2 - 10x - 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{2}{-2} = 1$$

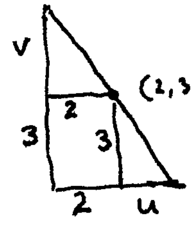
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1 - \frac{f(1)}{f'(1)}.$$ But $f'(1) = 0$, so this is undefined.

(c) (5 pts) Assuming you found that something goes wrong in part (b), what strategy would you suggest to fix it? Answer in one sentence without making any further calculations.

Start with x_1 inside the interval, for example $x_1 = \frac{1}{2}$.

4. (15 pts) Among all triangles with two sides lying along the x and y axes, and the third side passing through the point $(2, 3)$, find the one which has either the largest or the smallest area, and decide whether its area is in fact largest, or smallest.

Picture:



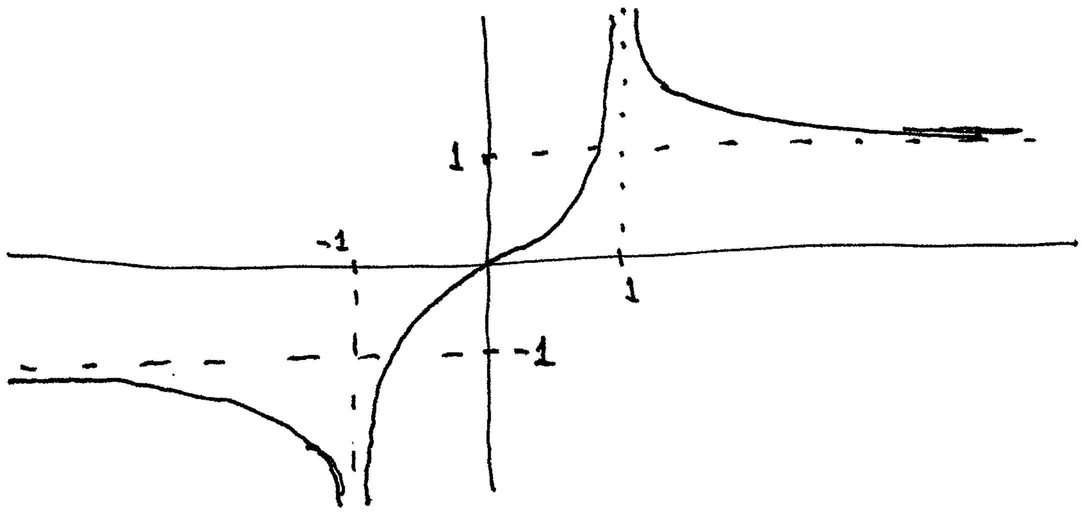
The two small triangles are congruent, so $v/2 = 3/u$, i.e. $v = 6/u$.
Area of the big triangle is

$A = \frac{1}{2}(u+2)(v+3)$, or $A(u) = \frac{1}{2}(u+2)(\frac{6}{u} + 3) = \frac{3}{2}u + \frac{6}{u} + 6$.

Then $A'(u) = \frac{3}{2} - \frac{6}{u^2} = 0$ when $\frac{3}{2} = \frac{6}{u^2}$, $u^2 = \frac{2}{3} \cdot 6 = 4$, $u = 2$
(the solution $u = -2$ is not relevant). The sign of $A'(u)$ is

$\ominus$, $\oplus$ (at $u=2$), so $A(u)$ has a minimum here. So the triangle has vertices $(0,0)$, $(4,0)$ and $(0,6)$, area 12, and this is the smallest possible area.

5. (15 pts) Suppose you know the following information about a function $f(x)$:
(a) $f(x)$ is an odd function, *i.e.*, $f(-x) = -f(x)$.
(b) $f(x)$ is continuous for all x except -1 and 1, and $\lim_{x\to 1} f(x) = +\infty$.
(c) $f'(x)$ is positive on $(-1, 1)$ and negative on $(1, \infty)$.
(d) $f''(x)$ is positive on $(0, 1)$ and $(1, \infty)$.
(e) $\lim_{x\to\infty} f(x) = 1$.
Sketch the graph of $f(x)$. Your graph should accurately reflect all the available information.



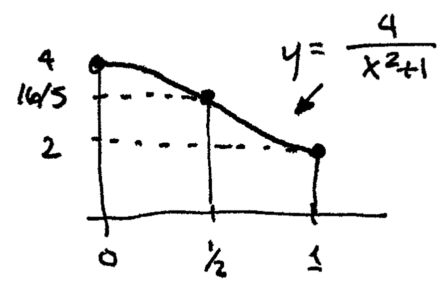
6. (a) (10 pts) Evaluate the integral

$$\int_0^1 \frac{4}{x^2+1}\, dx$$

$$\int_0^1 \frac{4}{x^2+1}\, dx = 4 \arctan(x) \Big]_0^1 = 4 \arctan(1) - 4 \arctan(0)$$
$$= 4 \cdot \frac{\pi}{4} - 4 \cdot 0 = \pi$$

(b) (10 pts) Use Riemann sum approximations to the integral in part (a), subdividing the interval $[0, 1]$ into two equal parts, to find rational numbers (*i.e.*, fractions with integer numerator and denominator) A and B such that

$$A < \pi < B.$$



Since $\frac{4}{x^2+1}$ is decreasing on $[0,1]$,
A = right endpoint Riemann sum
$= \frac{16}{5} \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{8}{5} + 1 = \frac{13}{5}$

B = left endpoint Riemann sum
$= 4 \cdot \frac{1}{2} + \frac{16}{5} \cdot \frac{1}{2} = \frac{18}{5}$