Math 1A — Fall 2010 — Haiman

HW 9 Solutions

4.4 42. $\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\sin x}{1/\ln x} = \lim_{x\to 0^+} \frac{\cos x}{-1/(x\ln^2 x)}$

$(0\cdot\infty)$ $(0/0)$

$= \lim_{x\to 0^+} -x\ln^2 x$ $(0\cdot\infty)$ $= \lim_{x\to 0^+} \frac{\ln^2 x}{1/x} = \lim_{x\to 0^+} \frac{2\ln x/x}{-1/x^2}$

(since $\cos x \to 1$) (∞/∞)

$= \lim_{x\to 0^+} -2x\ln x = \lim_{x\to 0^+} \frac{-2\ln x}{1/x} = \lim_{x\to 0^+} \frac{-2/x}{-1/x^2}$

$(0\cdot\infty)$ (∞/∞)

$= \lim_{x\to 0^+} 2x = \boxed{0}$

(or, simpler, $\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\sin x}{x}(x\ln x)$ and we know that $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$, $\lim_{x\to 0^+} x\ln x = 0$)

50. $\lim_{x\to 0} \cot x - \frac{1}{x} = \lim_{x\to 0}\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x\cos x - \sin x}{x\sin x}$ $(0/0)$

$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{-x\sin x}{\sin x + x\cos x}$ $(0/0)$

$= \lim_{x\to 0} \frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$

56. $\lim_{x\to\infty} (1+a/x)^{bx} = \exp\left(\lim_{x\to\infty} bx\ln(1+a/x)\right)$

(1^∞)

$\lim_{x\to\infty} bx\ln(1+a/x) = \lim_{x\to\infty} \frac{b\ln(1+a/x)}{1/x} = \lim_{x\to\infty} \frac{b/(1+a/x)\,(-a/x^2)}{-1/x^2}$

$(0\cdot\infty)$ $(0/0)$

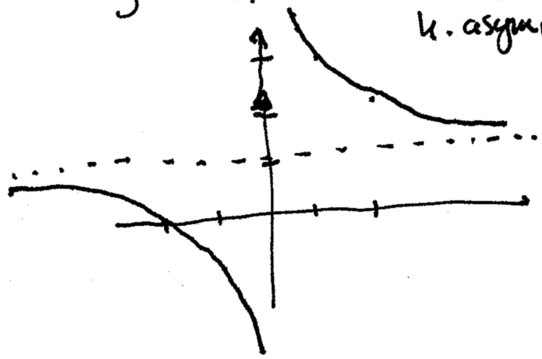
$= \lim_{x\to\infty} \frac{ab}{1+a/x} = ab$

Therefore $\lim_{x\to\infty} (1+a/x)^{bx} = e^{ab}$

4.5 10. $y = \dfrac{x^2-4}{x^2-2x} = \dfrac{(x+2)(x-2)}{x(x-2)} = \dfrac{x+2}{x} = 1+\dfrac{2}{x}$ (but $x=2$ is not in the domain)

$y = 1 + 2/x$ has v. asymptote $x=0$
h. asymptote $y=1$

$y' = -2/x^2$, so decreasing everywhere.
Crosses x axis at $x=-2$



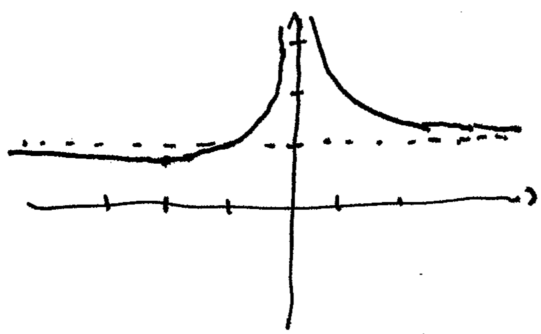
16. $y = 1 + \frac{1}{x} + \frac{1}{x^2} = \dfrac{x^2+x+1}{x^2}$

has v. asymptote $x=0$ (approached towards $+\infty$ from both sides), h. asymptote $y=1$

$y' = -\frac{1}{x^2} - \frac{2}{x^3} = -\dfrac{x+2}{x^3}$

decr.		incr.		decr.
$-$	-2	$+$	0	$-$

There is a local min $y(-2) = 3/4$



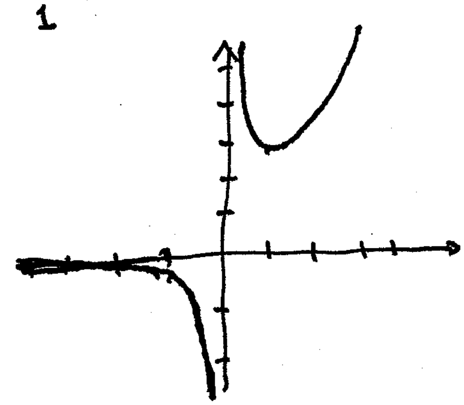
44. $y = e^x/x$ v. asymptote at $x=0$

h. asymptote $y=0$ as $x \to -\infty$
$y \to \infty$ as $x \to +\infty$

$y' = \left(\frac{1}{x} - \frac{1}{x^2}\right)e^x = \dfrac{x-1}{x^2}e^x$

decr.		decr.		incr.
$-$	0	$-$	1	$+$

local min $y(1) = e$
no intercepts on either axis



60. The leading terms $\frac{5x^4+\dots}{x^3+\dots}$ suggest a slant asymptote with slope 5, so lets compute

$$\lim_{x\to\pm\infty} y-5x = \lim_{x\to\pm\infty} \frac{5x^3+x^2-9x}{x^3-x^2+2} = \lim_{x\to\pm\infty} \frac{5+1/x-9/x^2}{1-1/x+2/x^3} = 5.$$

Therefore $y=5x+5$ is a slant asymptote, approached as $x\to\infty$ and as $x\to-\infty$.

4.8 12. Take $f(x)=x^{100}-100$, so $f'(x)=100x^{99}$, and the Newton iteration is $x_{i+1}=x_i-\frac{x_i^{100}-100}{100x_i^{99}} = x_i-\frac{x_i}{100}+x_i^{-99}$

If we start with $x_0=1$, we get

$x_1 = 1.99$
$x_2 \approx 1.9701$
$x_3 \approx 1.9504$
$x_4 \approx 1.9309$
$\vdots$

This seems not to be converging rapidly, but it's decreasing so let's try a new x_0 a bit bigger than 1, e.g.

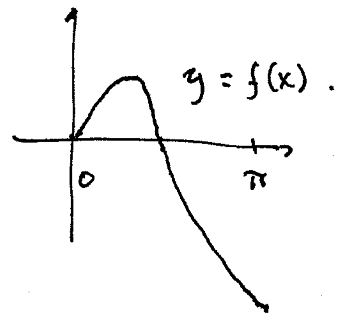
$x_0 = 1.1$
$x_1 \approx 1.08908$
$x_2 \approx 1.0784$
$x_3 \approx 1.06819$
$x_4 \approx 1.05896$
$\vdots$

after several more iterations it begins converging rapidly to $1.0471285480508 99\ldots$

16. Since $2\cos x \le 2$ we know the root of $2\cos x = x^4$ must be in $[0,\sqrt[4]{2}]$. Taking $x_0=1$, Newton's method with $f(x)=2\cos x - x^4$ has $x_2\approx x_3\approx 1.0139576\ldots$

As a check, $|2\cos x_3 - x_3^4|$ turns out to be $<10^{-13}$

36. $f'(x) = \cos x - x \sin x$



From the graph it's clear that the max of $f(x)$ occurs at the critical number c where $f'(x)=0$, somewhere in $[0, \pi/2]$.

Newton's method with $x_0 = 1$ gives

$$c \approx .860333.$$

The max is then $f(c) \approx .561096$