Math 1A - Fall 2010 - Haiman
HW 7 Solutions

4.2 12. $f(x) = x^3+x-1$ is continuous and differentiable everywhere. $\frac{f(2)-f(0)}{2-0} = 5$, so we are looking for $c \in (0,2)$ where $f'(c) = 5$. $f'(x) = 3x^2+1 = 5$ when $x^2 = 4/3$, $x = \pm 2\sqrt{3}/3$. The negative solution is not in $(0,2)$ so $c = 2\sqrt{3}/3$.

18. Let $f(x) = 2x-1-\sin x$. Since $f(0) = -1$, $f(\pi) = 2\pi - 1 > 0$, and f is continuous, $f(x) = 0$ has at least one root in $(0, 2\pi)$ by Intermediate Value Theorem.
$f'(x) = 2 - \cos x > 0$ for all x. By Rolle's theorem, any two roots must be separated by a root of $f'(x) = 0$, so there is only one root.

20. If $f(x) = x^4+4x+c$, then $f'(x) = 4x^3+4$, with $f'(x) = 0$ at $x = -1$ only. Since f is differentiable and $f'(x) = 0$ has just one root, Rolle's theorem implies that $f(x) = 0$ has at most two roots.

24. By MVT, $f(8)-f(2) = f'(c)(8-2)$ for some $c \in (2,8)$. By assumption, $3 \le f'(c) \le 5$, hence $3 \cdot 6 \le f(8)-f(2) \le 5 \cdot 6$

26. $h(x) = f(x)-g(x)$ is continuous on $[a,b]$ and differentiable on (a,b), so MVT applies to show $h(b)-h(a) = h'(c)(b-a)$ for some $c \in (a,b)$. Now $h(a) = f(a)-g(a) = 0$, and $h'(c) = f'(c)-g'(c) < 0$ for every $c \in (a,b)$.
Hence $h(b) = h'(c)(b-a) < 0$, i.e. $f(b)-g(b) < 0$, so $f(b) < g(b)$.

4.3 2. (a) $(0,1)$ and $(3,7)$

(b) $(1,3)$

(c) $(2,4)$ and $(5,7)$

(d) $(0,2)$ and $(4,5)$

(e) $(2,2)$, $(4,3)$, $(5,4)$

12. (a) $$f'(x) = \frac{(x^2+3)2x - x^2(2x)}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$

On $(-\infty,0)$, $f'(x)<0$, f is decreasing

on $(0,\infty)$, $f'(x)>0$, f is increasing

(b) Local min $f(0)=0$, by 1st derivative test.
No others, since $c=0$ is the only critical number

(c) $$f''(x) = \frac{-18(x^2-1)}{(x^2+3)^3} = \frac{-18(x-1)(x+1)}{(x^2+3)^3}$$

on $(-\infty,-1)$ and $(1,\infty)$, $f''(x)<0$, concave down

on $(-1,1)$, $f''(x)>0$, concave up

14. (a) $f'(x) = -2\cos x \sin x - 2\cos x = -2\cos x \overset{(\geq 0)}{(1+\sin x)}$

On $(0, \pi/2)$ and $(3\pi/2, 2\pi)$, $f'(x)<0$, f decreasing

On $(\pi/2, 3\pi/2)$, $f'(x)>0$, f increasing

b) Local min $f(\pi/2) = -2$
Local max $f(3\pi/2) = 2$
No other critical numbers

(endpoints $0, 2\pi$ of the domain do not count as local extrema, according to the definition used in our textbook)

c) $f''(x) = 2\underset{(\geq 0)}{(\sin x+1)}(2\sin x - 1)$

on $(0, \pi/6)$ and $(5\pi/6, 2\pi)$, $f''(x)<0$, concave down

on $(\pi/6, 5\pi/6)$, $f''(x)>0$, concave up

44. $f(t) = t + \cos t$ on $[-2\pi, 2\pi]$

$f'(t) = 1 - \sin t$

$f''(t) = -\cos t$

a) Increasing on the whole interval

b) No local max, min

c) Concave up on $(-3\pi/2, -\pi/2)$ and $(\pi/2, 3\pi/2)$

down on $(-2\pi, -3\pi/2)$, $(-\pi/2, \pi/2)$, and $(3\pi/2, 2\pi)$

Inflection points:

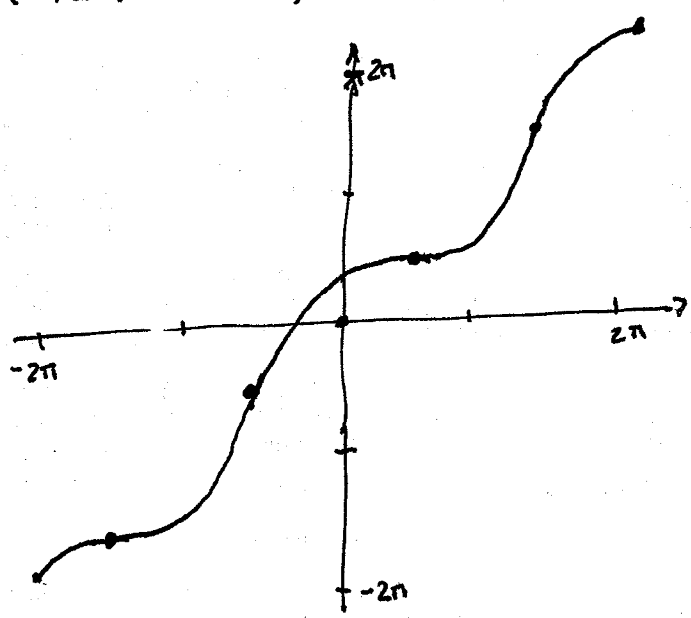
$(-3\pi/2, -3\pi/2)$, slope $f'(-3\pi/2) = 0$

$(-\pi/2, -\pi/2)$, slope $f'(-\pi/2) = 2$

$(\pi/2, \pi/2)$, slope $f'(\pi/2) = 0$

$(3\pi/2, 3\pi/2)$, slope $f'(3\pi/2) = 2$

d)



50. $f(x) = e^x/(1+e^x) = 1/(e^{-x}+1) = 1 - \frac{1}{e^x+1}$

$f'(x) = e^x/(1+e^x)^2$

$f''(x) = e^x(1-e^x)/(1+e^x)^3$

$\lim_{x\to\infty} f(x) = \frac{1}{0+1} = 1 \qquad \lim_{x\to-\infty} f(x) = \frac{0}{1+0} = 0$

a) Hor. asymptotes $y = 1$, approached from below as $x \to \infty$
$y = 0$, " " above as $x \to -\infty$

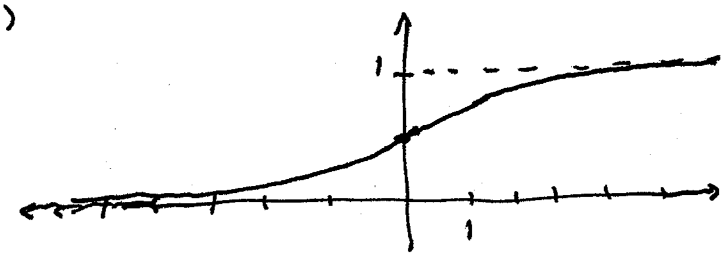
No vertical asymptotes

b) Increasing everywhere

c) No local min or max

d) Concave up on $(-\infty, 0)$, down on $(0, \infty)$
Inflection point $(0, \tfrac{1}{2})$ with slope $\tfrac{1}{4}$

e)



70. For $y = e^{-x}\sin x$, we have

$y' = e^{-x}(\cos x - \sin x)$

$y'' = -2e^{-x}\cos x$.

So y'' changes sign when $\cos x$ does: at $x = \ldots, -\frac{3\pi}{2}, -\pi/2, \pi/2, 3\pi/2, \ldots$
i.e. $\pi/2 + k\pi$ for integers k.

The inflection points are $x = \frac{\pi}{2} + k\pi$, $y = e^{-(\pi/2 + k\pi)} \cdot \begin{cases} +1 & \text{if } k \text{ even} \\ -1 & \text{if } k \text{ odd} \end{cases}$

So the inflection points $x = \pi/2 + k\pi$: k even
are also on the curve $y = e^{-x}$,
while those with k odd are on $y = -e^{-x}$.
The curves just touch at these points, without crossing,
because $-e^{-x} \le e^{-x}\sin x \le e^{-x}$ for all x.

76 a) Consider the difference, $y = e^x - (1+x)$.

$y' = e^x - 1 \geq 0$ for $x \geq 0$, and $y(0) = 0$.

So, since $y(0) = 0$ and y is increasing for $x \geq 0$, we have $y \geq 0$ for $x \geq 0$. Hence $e^x \geq 1+x$.

b) Consider $z = e^x - (1 + x + x^2/2)$.

$z' = e^x - (1+x) \geq 0$ for $x \geq 0$ by part (a).

Again $z(0) = 0$, and z is increasing for $x \geq 0$, so $z \geq 0$ for $x \geq 0$. Hence $e^x \geq 1 + x + x^2/2$.