Prof. Haiman Math 1A—Calculus Fall, 2006

Final Examination Solutions

1. Simplify $x^{1/\ln x}$.

$$x^{1/\ln x} = e^{\ln x/\ln x} = e.$$

2. If $f(x)$ is continuous on $[0, 2]$, and $f(0) = 1$, $f(1) = 2$, $f(2) = 0$, show that f is not one-to-one.

Pick a number between 1 and 2, say 3/2. By the Intermediate Value Theorem we must have $f(x) = 3/2$ for some $x \in (0, 1)$ and also for some $x \in (1, 2)$, so f is not one-to-one.

3. Find the equation of the tangent line to $x^3 + y^3 = 9$ at $(2, 1)$.

Differentiate to get $3x^2 + 3y^2y' = 0$. At $(2, 1)$ this gives $y' = -4$. The tangent line is therefore $y = -4(x - 2) + 1 = -4x + 9$.

4. Evaluate the limit (as a number or an infinite limit):

$$\lim_{x\to\pi/2} \frac{1 - \sin x}{\cos^2 x}$$

This has the "0/0" form. L'Hospital gives

$$\lim_{x\to\pi/2} \frac{1 - \sin x}{\cos^2 x} = \lim_{x\to\pi/2} \frac{-\cos x}{-2\cos x \sin x} = \lim_{x\to\pi/2} \frac{1}{2\sin x} = 1/2.$$

5. Evaluate the limit (as a number or an infinite limit):

$$\lim_{x\to+\infty} (1 + 2/x)^x$$

Observe that $(1 + 2/x)^x = e^{x\ln(1+2/x)}$. Now

$$\lim_{x\to\infty} x\ln(1 + 2/x) = \lim_{x\to\infty} \ln(1 + 2/x)/(1/x).$$

The last expression has "0/0" form, and is equal by L'Hospital's rule to

$$\lim_{x\to\infty} \frac{-2/(x^2(1 + 2/x))}{-1/x^2} = \lim_{x\to\infty} 2/(1 + 2/x) = 2.$$

Therefore, $\lim_{x\to+\infty}(1 + 2/x)^x = e^2$.

6. Find c such that the line $y = x + c$ is a slant asymptote to the curve $y = x^2/(x + 5)$.

The required c is given by the limit (since it exists)

$$c = \lim_{x\to\infty} \Big(\frac{x^2}{x+5} - x\Big) = \lim_{x\to\infty} \frac{x^2-(x^2+5x)}{x+5} = \lim_{x\to\infty} \frac{-5x}{x+5} = -5.$$

7. Find $(d/dx)^{17}(e^x + e^{-x})$.

The higher derivatives alternate between $e^x + e^{-x}$ and $e^x - e^{-x}$. Since 17 is odd, $(d/dx)^{17}(e^x + e^{-x}) = e^x - e^{-x}$.

8. If X and Y are functions of t related by $Y = e^{XY}$, find X' when $Y = 1$ and $Y' = 3$.

Differentiate to get $Y' = e^{XY}(X'Y + XY')$. When $Y = 1$, we have $1 = e^{1X}$, so $X = 0$. Substitute $Y = 1$, $Y' = 3$, $X = 0$ into $Y' = e^{XY}(X'Y + XY')$ to get $X' = 3$.

9. Find the point on the line $x + 2y = 3$ closest to the origin.

It's easiest to minimize the square of the distance from the origin, which is $x^2 + y^2$. Use the equation $x + 2y = 3$ to express this in terms of y as $(3 - 2y)^2 + y^2 = 5y^2 - 12y + 9$. Set the derivative to zero to find the minimum at $10y - 12 = 0$, $y = 6/5$, $x = 3 - 12/5 = 3/5$. So the closest point is $(3/5, 6/5)$.

10. Find all local minima and maxima of the function $f(x) = x^2e^{-x}$, and the intervals where f is increasing or decreasing.

Differentiate to get $f'(x) = x(2 - x)e^{-x}$. Therefore $f(x)$ is decreasing on $(-\infty, 0]$ and $[2,\infty)$, and increasing on $[0, 2]$ with a local (and absolute) minimum at $x = 0$, $f(0) = 0$, and a local (but not absolute) maximum at $x = 2$, $f(2) = 4e^{-2}$.

11. Show that the equation $x^3 - 3x + 3 = 0$ has exactly one real root.

Set $f(x) = x^3 - 3x + 3$. Then $f'(x) = 3x^2 - 3$. Therefore $f(x)$ is decreasing on $[-1, 1]$ and increasing on $[1,\infty)$, hence $f(1) = 1$ is a minimum on $[-1,\infty)$. This shows that there is no root in $[-1,\infty)$. Furthermore, $f(x)$ is increasing on $(-\infty, -1]$, and therefore has at most one root. Since $f(-1) = 5 > 0$ and (for instance) $f(-3) = -15 < 0$, there is a root in the interval $(-3, -1)$.

12. Using Newton's method to find an approximate solution to the equation $x^3 = 2$, starting with first approximation $x_1 = 1$, find the next approximation.

We are finding a zero of $f(x) = x^3 - 2$. Its derivative is $f'(x) = 3x^2$. The Newton step is $x_2 = x_1 - f(x_1)/f'(x_1) = 1 - (-1)/3 = 4/3$.

13. Find $f(x)$ such that $f''(x) = 1 + \sin x$, $f(0) = 0$, and $f'(0) = 0$.

Antidifferentiate once to get $f'(x) = x - \cos x + C$. Use $f'(0) = 0$ to see that $C = 1$. Antidifferentiate again to get $f(x) = x^2/2 + x - \sin x + D$. Use $f(0) = 0$ to see that $D = 0$, so $f(x) = x^2/2 + x - \sin x$.

14. Show that $\int_0^1 e^{-x^2}\, dx \le (1 + e^{-1/4})/2$.

$$\int_0^1 e^{-x^2}\,dx = \int_0^{1/2} e^{-x^2}\,dx + \int_{1/2}^1 e^{-x^2}\,dx.$$

Since e^{-x^2} is a decreasing function on $[0,1]$, the maximum value of the integrand occurs at the left limit of integration in each integral, giving

$$\int_0^{1/2} e^{-x^2}\,dx + \int_{1/2}^1 e^{-x^2}\,dx \le e^0/2 + e^{-1/4}/2 = (1+e^{-1/4})/2.$$

15. Differentiate the function $F(x) = \int_1^{1/x} \sin^{-1}(t)\,dt$

We have $F(x) = G(1/x)$ where $G'(x) = \sin^{-1}(x)$ by the Fundamental Theorem of Calculus. By the Chain Rule,

$$F'(x) = G'(1/x)(-1/x^2) = -\frac{\sin^{-1}(1/x)}{x^2}.$$

16. Evaluate the integral $\int_{-1}^2 |x^3|\,dx$.

Note that $x^3 \le 0$ on $[-1,0]$ and $x^3 \ge 0$ on [0,2]. Therefore

$$\int_{-1}^2 |x^3|\,dx = \int_{-1}^0 -x^3\,dx + \int_0^2 x^3\,dx = -\frac{x^4}{4}\Big]_{-1}^0 + \frac{x^4}{4}\Big]_0^2 = \frac{1}{4} + 4 = \frac{17}{4}.$$

17. Evaluate the integral $\int_1^e \sqrt{\ln x}/x\,dx$.

Let $u = \ln x$ to get

$$\int_0^1 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big]_0^1 = \frac{2}{3}.$$

18. Find the area of the region enclosed by the line $x = 1$ and the curves $8y = x^2$ and $xy = 1$.

Solve $8y = x^2$ and $xy = 1$ together to locate the right endpoint of the region at $(2, 1/2)$. The area is given by

$$\int_1^2 \frac{1}{x} - \frac{x^2}{8}\,dx = \ln x - \frac{x^3}{24}\Big]_1^2 = \ln 2 - \frac{7}{24}.$$

19. Find the volume of the solid of rotation about the y-axis of the region in the first quadrant enclosed by the y-axis, the line $y = x + 1$, and the curve $y = 2x^2$.

This is most easily done using cylindrical shells. The two curves cross at $(1, 2)$, so the volume is

$$2\pi \int_0^1 (x + 1 - 2x^2)x \, dx = 2\pi \left[\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{2} \right]_0^1 = \frac{2\pi}{3}.$$

To calculate the same thing using slices, you have to do two integrals, one for y from 0 to 1 and another for y from 1 to 2.

20. For the function $f(x) = 1/x$, find the point c in the interval $(1, 3)$ such that $f(c)$ is equal to the average value of f on the interval $[1, 3]$.

The average value is

$$f_{\text{av}} = \frac{1}{2} \int_1^3 \frac{dx}{x} = \frac{\ln 3}{2}.$$

The required c satisfies $f(c) = 1/c = f_{\text{av}}$, therefore $c = 2/(\ln 3)$.