Prof. Haiman | Math 1A—Calculus | Fall, 2006

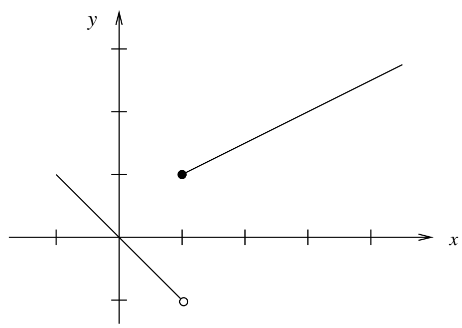
First Midterm Exam Solutions

Name ____________________ Student ID Number ____________________

Section time and Instructor ____________________

You may use one sheet of notes. No other notes, books or calculators allowed. There are 10 questions, on front and back. Write answers on the exam and turn in only this paper. Show enough work so that we can see how you arrived at your answers.

1. Write a formula for the function whose graph is shown. Assume the lines continue to infinity outside the part of the graph shown here, and that their slopes are simple fractions.



$$f(x) = \begin{cases} -x & x < 1, \\ (x+1)/2 & x \geq 1 \end{cases}$$

2. If $f(x) = 2x$, $g(x) = 1/x$, and $h(x) = x + 5$, find $f \circ g \circ h$.

$$f \circ g \circ h(x) = \frac{2}{x+5}.$$

3. Which of the following are 1-1 functions?

(a) $f(x) = x^3$, for all real numbers x

(b) $f(x) = x^4$, for all real numbers x

(c) $f(x) = x^3$, for $x \geq 0$

(d) $f(x) = x^4$, for $x \geq 0$

All except (b).

4. Find the inverse function of $f(x) = 3\sin(x + (\pi/4))$.

$f^{-1}(x) = \sin^{-1}(x/3) - \pi/4$.

5. Simplify $16^{\log_2(x)}$.

$16^{\log_2(x)} = 2^{4\log_2(x)} = 2^{\log_2(x^4)} = x^4$. Strictly speaking, this is valid only for $x > 0$.

6. Find the equations of the horizontal and vertical asymptotes to the graph of $f(x) = (x+1)(x+3)/((x+2)(x+4))$.

Since $f(x)$ has infinite limits at $x \to -2^{\pm}$, $x \to -4^{\pm}$, the lines $x = -2$ and $x = -4$ are vertical asymptotes. Since $\lim_{x\to\pm\infty} f(x) = 1$, the line $y = 1$ is a horizontal asymptote (in both directions).

7. Evaluate the limit

$$\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}.$$

$$\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} = \lim_{x\to 3}(\sqrt{x}+\sqrt{3}) = 2\sqrt{3}.$$

8. Show that the equation $2^x = x + 3$ has a solution in the interval $2 < x < 3$.

Let $f(x) = 2^x - x$. The function f is continuous on $[2,3]$, and $f(2) = 2$, $f(3) = 5$. By the Intermediate Value Theorem, there must be some x in $[2,3]$ such that $f(x) = 3$, which means x is a solution of the given equation. Clearly the solution is not $x = 2$ or $x = 3$, so it lies in the open interval $(2,3)$.

9. Find the equation of the tangent line to the graph $y = 3x/(x+1)$ at the point $(2,2)$.

Differentiating, $y' = 3/(x+1)^2$. The slope of the tangent line is given by $y'(2) = 1/3$. The equation of the line is $y = (1/3)(x-2) + 2$.

10. Use the definition of derivative to evaluate $f'(x)$ for $f(x) = 1/x^2$, thereby verifying the power rule for $n = -2$.

Working from the definition, we calculate the limit

$$\lim_{u\to x} \frac{1/u^2 - 1/x^2}{u-x} = \lim_{u\to x} \frac{x^2-u^2}{u^2x^2(u-x)} = \lim_{u\to x} \frac{-(x+u)}{u^2x^2} = -2x/x^4 = -2/x^3.$$