Math 1A, Calculus **Final Exam Solutions** Haiman, Fall 2004

1. Evaluate the limit if it exists (possibly as an infinite limit).

$$\text{(a)} \quad \lim_{x\to 1} \frac{1}{\ln x} \qquad\qquad \text{(b)} \quad \lim_{x\to 1} \frac{1}{(\ln x)^2}$$

(a) $\lim_{x\to 1} 1/\ln x$ does not exist, (b) $\lim_{x\to 1} 1/(\ln x)^2 = +\infty$.

2. Differentiate the function $y = \sin(\sin(\sin x))$.

$$y' = \cos(\sin(\sin x))\cos(\sin x)\cos x.$$

3. Find (a) all local maxima and minima of the function

$$f(x) = \frac{x}{x^2+1},$$

and (b) the intervals of increase or decrease of $f(x)$.

(a) $f'(x) = (1-x^2)/(x^2+1)^2$ is zero at $x = \pm 1$. $f(-1) = -1/2$ is a local (and absolute) minimum, and $f(1) = 1/2$ is a local (and absolute) maximum. (b) $f(x)$ is decreasing on $(-\infty, -1)$ and $(1, \infty)$, increasing on $(-1, 1)$.

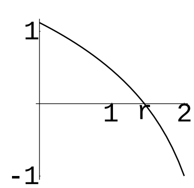
4. Find the linear approximation to the function $f(x) = \ln x$ near $x = 2$.

Since $f'(2) = 1/2$, $f(2) = \ln 2$, the linear approximation is $y = (x-2)/2 + \ln 2$.

5. If $y = e^{xy}$, express dy/dx in terms of x and y.

$$\begin{aligned} y' &= e^{xy}(xy' + y) \\ (1 - xe^{xy})y' &= ye^{xy} \\ y' &= \frac{ye^{xy}}{1 - xe^{xy}}. \end{aligned}$$

6. Suppose we use Newton's method to approximate the root r of the function whose graph is shown, using $x_1 = 1$ for the first approximation.



For the next approximation x_2, decide whether $x_2 < r$ or $x_2 > r$. Justify your answer.

The tangent line at $x = 1$ crossess the x axis to the right of r, because the graph is concave downward. Therefore $x_2 > r$.

7. Find the largest area of a rectangle with horizontal and vertical sides, lower-left corner at the origin $(0,0)$, and upper-right corner on the curve $y = e^{-x}$.

We must maximize $A = xe^{-x}$. We have $dA/dx = (1-x)e^{-x} = 0$ at $x = 1$. It's a maximum by the first derviative test. The area is $A = e^{-1}$.

8. Find the limit.

$$\lim_{x\to\infty} x^{1/(1+\ln x)}$$

We have $\lim_{x\to\infty} x^{1/(1+\ln x)} = \lim_{x\to\infty} e^{(\ln x)/(1+\ln x)}$. Now $\lim_{x\to\infty}(\ln x)/(1+\ln x) = 1$, so $\lim_{x\to\infty} x^{1/(1+\ln x)} = e$.

9. If $\int_a^x f(t)\,dt = x\ln x$ for all $x > 0$, find the function $f(x)$ and the constant a.

By the fundamental theorem of calculus, $f(x) = \frac{d}{dx}(x\ln x) = 1 + \ln x$. Since $x\ln x = 0$ at $x = 1$, the constant a is equal to 1.

10. Evaluate the integral.

$$\int_0^2 xe^{-x^2}\,dx$$

Substitute $u = -x^2$, $du = -2x\,dx$, to get

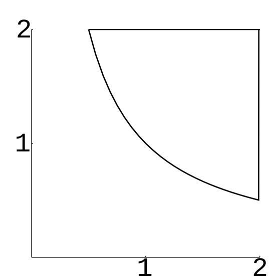
$$-\frac{1}{2}\int_0^{-4} e^u\,du = -\frac{1}{2}e^u\Big]_0^{-4} = \frac{1-e^{-4}}{2}.$$

11. Evaluate the indefinite integral.

$$\int \frac{(x+1)(x+2)}{x^2}\,dx$$

$$\int \frac{(x+1)(x+2)}{x^2}\,dx = \int 1 + 3x^{-1} + 2x^{-2}\,dx = x + 3\ln x - 2/x + C.$$

12. Sketch the region enclosed by the lines $x = 2$, $y = 2$ and the curve $xy = 1$, and find its area.



The area is given by

$$\int_{1/2}^2 2 - 1/x\,dx = 2x - \ln x\Big]_{1/2}^2 = 3 - \ln 2 + \ln(1/2) = 3 - 2\ln 2.$$

13. Find the average value of the function $f(x) = 1/x$ on the interval $[1, 3]$.

$$\frac{1}{2}\int_1^3 \frac{dx}{x} = \frac{\ln x}{2}\Big]_1^3 = \frac{\ln 3}{2}.$$

14. Find the volume of the circular cone obtained by rotating the triangle enclosed by the x and y axes and the line $x + y = 1$ about the y axis.

$$\int_0^1 \pi(1-y)^2\, dy = -\int_1^0 \pi u^2 du = -\pi u^3/3\Big]_1^0 = \pi/3.$$

15. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region enclosed by the x axis, the line $x = 2$, and the curve $y = xe^{-x}$ about the y axis.

$$\int_0^2 2\pi x^2 e^{-x}\, dx.$$