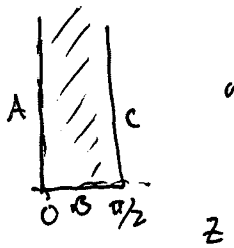
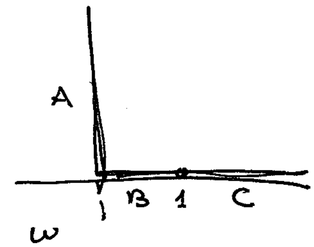
PS9 Solutions

121.2 The transformation $w = \sin z$ maps the strip



onto the first quadrant :



In the w plane, $h(u,v) = \frac{2}{\pi}\mathrm{Arg}(w)$ solves the problem : it is 0 on the x-axis, 1 on the y-axis and harmonic because $\mathrm{Arg}(w) = \mathrm{Im}(\mathrm{Log}(w))$. In the z plane this gives $H(x,y) = \frac{2}{\pi}\mathrm{Arg}(\sin z)$. To express it in terms of x and y, use $\sin z = \sin x \cosh y + i \cos x \sinh y$ to see that

$$\mathrm{Arg}(\sin z) = \tan^{-1}\frac{\cos x \sinh y}{\sin x \cosh y} = \tan^{-1}\left(\frac{\tanh y}{\tan x}\right),$$

at least when its values are in $(-\pi/2, \pi/2)$, as they are in this problem.

Thus $H(x,y) = \frac{2}{\pi}\tan^{-1}\left(\frac{\tanh y}{\tan x}\right)$.

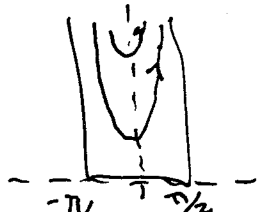
126.4 The velocity field (using complex numbers to represent the velocity vectors) is $\overline{F'(z)} = A(1 - 1/\bar{z}^2)$. On the unit circle $|z|=1$, we have $\bar{z} = 1/z$ and can write this as $A(1-z^2) = Az\left(\frac{1}{z} - z\right) = Az(\bar{z} - z) = -2iAz\,\mathrm{Im}(z)$. The speed is the norm of this, $|-2iAz\,\mathrm{Im}(z)| = 2A|y|$ (since $|z|=1$).
$= 2A|\sin\theta|$.

By Bernoulli's equation, the pressure is greatest where the speed is smallest, at $\sin\theta = 0$, or $z = \pm 1$. The pressure is smallest where the speed is largest, at $|\sin\theta| = 1$, or $z = \pm i$.

[Once again I've put a misprint on the problem set — as you may have deduced from the hint and the fact that we did 126.4 in class, I meant to assign 126.6. Sorry.]

126.7 $w = \sin z$ maps the region to the upper half-plane, where we have an obvious solution $F = Aw$ with ~~constant~~ velocity field $\overline{F'(z)} = A$ (A is real), representing flow to the right at constant velocity A.

Then $F = A \sin z$ is the corresponding complex potential for flow in the region



The stream function is $\operatorname{Im}(A \sin z) = A \cos x \sinh y$, so the streamlines have equation $\cos x \sinh y = C$, or $y = \sinh^{-1}(C \sec x)$.
Notice that $\sec x > 0$ on $(-\pi/2, \pi/2)$ and $\to \infty$ as $x \to \pm \pi/2$, while $\sinh^{-1}(t)$ is increasing from 0 on $[0, \infty)$. So the relevant valules of the constant C are $C > 0$, for which the graphs of $y = \sinh^{-1}(C \sec x)$ are above the x-axis and look like the streamlines drawn above.