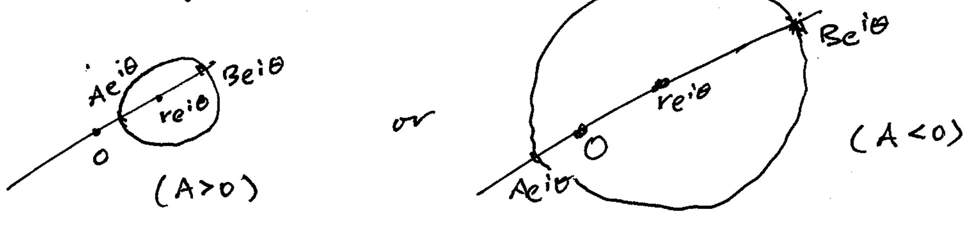
PS 8 Solutions

98.12 If the original circle has center at $z=0$, then it is mapped to another circle centered at $z=0$, but the center is not mapped to the center, since $w = 1/z$ is undefined at $z=0$.

Otherwise, say the original circle is centered at $re^{i\theta}$, where $r>0$. Then the points on the original circle farthest from and closest to the origin are at ~~$Be^{i\theta}$~~ $Ae^{i\theta}$, $Be^{i\theta}$, where $A > r$ and $|B| < A$:



These are related by $r = (A+B)/2$. The new circle has farthest point $\frac{1}{A}e^{-i\theta}$ and nearest point $\frac{1}{B}e^{-i\theta}$, with center at $r'e^{-i\theta}$, where $r' = (\frac{1}{A} + \frac{1}{B})/2$. Note that $r'<0$ in the case $A<0$, so the actual distance from $z=0$ to the new center is $|r'|$ in that case.

~~It is enough to show that $r' > r$ in the case $A \gtrless 0$, and $r' \neq r$ in the case $A>0$.~~

The original center goes to $\frac{1}{r}e^{-i\theta}$, so we need to show that $r' \neq 1/r$, i.e. $rr' \neq 1$. Now,

$$rr' = \frac{1}{4}\left(\frac{A}{B} + \frac{B}{A} + 2\right).$$

If $A<0$, then A/B and B/A are both negative, so $rr' < 1/2$.

If $A>0$, then ~~$rr' \geq \frac{1}{4}(x+x^{-1})^2$, where $x \geq \sqrt{B/A}$~~

$$rr' - 1 = \frac{1}{4}\left(\frac{A}{B} + \frac{B}{A} - 2\right) = \frac{1}{4}(x - x^{-1})^2, \text{ where } x = \sqrt{B/A},$$

So $rr' > 1$, since $B > A$, so $x > 1$, so $x \neq x^{-1}$.

100.2 The transformation $s = \dfrac{z+i}{2z}$ maps $(-i, 0, i)$ to $(0, \infty, 1)$.

The transformation $s = \dfrac{1-i}{2}\,\dfrac{w+1}{w-i}$ maps $(-1, i, 1)$ to $(0, \infty, 1)$.

Inverting the second of these:

$$s(w-i) = \frac{1-i}{2}(w+1)$$

$$w\left(s + \frac{i-1}{2}\right) = \frac{1-i}{2} + is$$

$$w = \frac{is + \frac{1-i}{2}}{s - \frac{1-i}{2}} = \frac{(-1+i)s+1}{(1+i)s-1}$$

Then composing with the first:

$$w = \frac{(-1+i)\frac{z+i}{2z}+1}{(1+i)\frac{z+i}{2z}-1} = \frac{(-1+i)(z+i)+2z}{(1+i)(z+i)-2z} = \frac{(1+i)z-1-i}{(-1+i)z-1+i}$$

$$= -i\,\frac{z-1}{z+1}$$

The imaginary axis must map onto a circle or line containing -1, i, and 1, hence onto the unit circle $|w|=1$.

100.6 Suppose $w=f(z)$ had three fixed points $f(z_1)=z_1$, $f(z_2)=z_2$, $f(z_3)=z_3$. Since a Möbius transformation is determined by the images of any 3 points, f must be the identity.

100.7 (a) For $w = \dfrac{z-1}{z+1}$, we have $\dfrac{i-1}{i+1} = i$, $\dfrac{-i-1}{-i+1} = -i$, and these two are all the fixed points, by 100.6

(b) If z_0 is a fixed point of $w = \dfrac{6z-9}{z}$, then $z_0 = \dfrac{6z_0-9}{z_0}$,

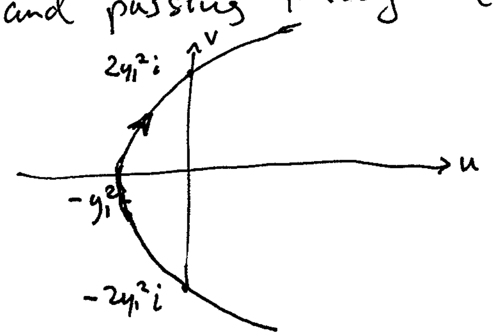
$z_0^2 = 6z_0 - 9$, $z_0-6z_0+9 = (z_0-3)^2 = 0$, so $z_0 = 3$ is the unique fixed point (other than ∞, but $z=\infty$ maps to $w=6$, so ∞ is not a fixed point).

100.12 Since $T(z) = \dfrac{az+b}{cz+d}$ maps $(0, 1, \infty)$ to $\left(\frac{b}{d}, \frac{a+b}{c+d}, \frac{a}{c}\right)$, its inverse is $T^{-1}(z) = \dfrac{z-b/d}{z-a/c}\cdot\dfrac{\frac{a+b}{c+d}-a/c}{\frac{a+b}{c+d}-b/d} = \dfrac{-dz+b}{cz-a}$. From this it is obvious that if $d=-a$ then $T^{-1}(z) = T(z)$. Conversely, if $T^{-1}(z) = T(z)$, then there must be a constant s such that $a=-sd$, $b=sb$, $c=sc$, $d=-sa$. But then $s=1$, so $d=-a$.

102.7 The condition $z \in \mathbb{C} - \mathbb{R}_{\ge 1}$ is the same as $z-1 \in \mathbb{C} - \mathbb{R}_{\ge 0}$, i.e. $z \in \mathbb{C} - \mathbb{R}_{\ge 0}$. We need $\log(z)$ to map this onto $0 < v < 2\pi$, so we must take the branch $\log(z) = \ln|z| + i \arg z$ with $0 < \arg z < 2\pi$.

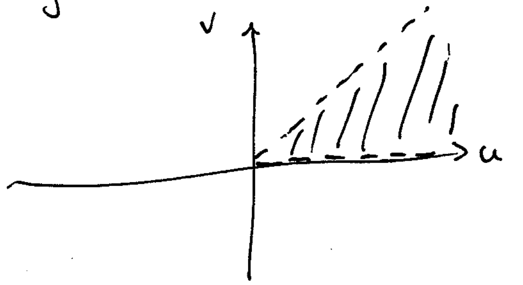
106.9 In §105 we saw that $\sin z$ maps horizontal segments from $-\pi/2 + iy$ to $\pi/2 + iy$ onto the upper halves of ellipses with foci at ± 1. Since $\sin(\pi + z) = -\sin z$, it follows that the segment from $\pi/2 + iy$ to $\pi + iy$ maps to the quarter of the ellipse in the 4th quadrant, and $-\pi + iy$ to $-\pi/2 + iy$ to the quarter in the 3rd quadrant. Thus the segment $-\pi + iy$ to $\pi + iy$ maps onto the whose ellipse, with starting and ending point on the negative imaginary axis. Applying this to all segments with $a \le y \le b$, for $0 < a < b$, we see that the rectangle $-\pi \le x \le \pi$, $a \le y \le b$ maps onto the region between the two ellipses for $y = a$ and $y = b$, and the mapping is one-to-one except on the vertical sides, from $-\pi + ia$ to $-\pi + ib$, and from $\pi + ia$ to $\pi + ib$, both of which map onto the segment of the imaginary axis between $\sin(\pm\pi + ia) = -\sin(ia) = -i\sinh(a)$ and $\sin(\pm\pi + ib) = -i\sinh(b)$.

108.1 The line $z = x + iy_1$ maps to $w = z^2 = x^2 - y_1^2 + 2iy_1x$, i.e. $u = x^2 - y_1^2$, $v = 2y_1x$. These are parametric equations equivalent to $u = \left(\frac{v}{2y_1}\right)^2 - y_1^2$, or $v^2 = 4y_1^2(u + y_1^2)$, if $y_1 \ne 0$, which describes a parabola symmetric about the u axis, opening to the right, and passing through $(-y_1^2, 0)$ and $(0, \pm 2y_1^2)$:

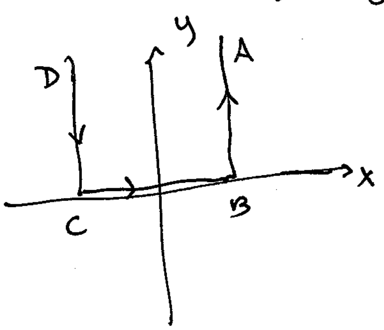


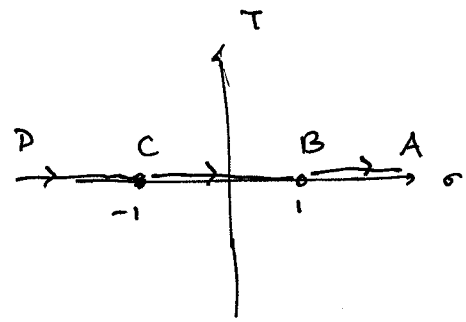
For $y_1 > 0$, v increases with x, so the line $y = y_1$ oriented to the right $\rightarrow$ maps to the parabola oriented as shown. The focus is a 0 because the segment between the v intercepts is 4 times the length of the segment between 0 and the u intercept.

108.6 (This was a misprint. I actually meant to assign 108.4).
We know from §105 that $s = \sin z$ maps $\{-\pi/2 < x < \pi/2,\ y>0\}$ onto the upper half plane $\operatorname{Im} s > 0$, or equivalently ~~arg~~ $0 < \arg(s) < \pi$.
Then, using the principal branch of $w = s^{1/4}$, this goes to $0 < \arg(w) < \pi/4$, which is the sector between the u axis and the ray $u = v \geq 0$ in the w plane:

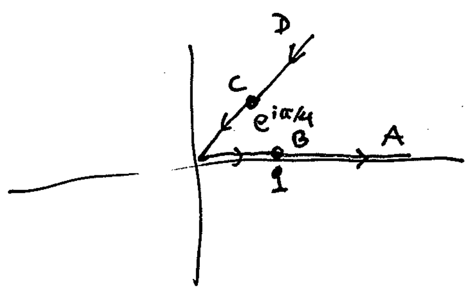


The ~~boundary~~ segments map first into the s plane as

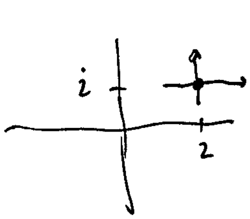




and then under $w = s^{1/4}$ to

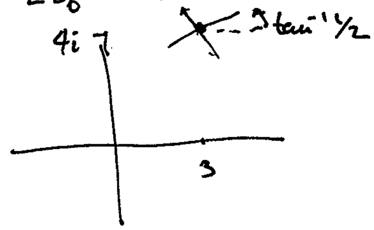


114.1 $w' = 2z$, and $2z_0 = 4 + 2i = 2\sqrt{5}\, e^{i \tan^{-1} \frac{1}{2}}$. So the local angle of rotation is $\tan^{-1} \frac{1}{2}$ and the scale factor is $2\sqrt{5}$.



z

$z_0 = 2 + i$



$w = z^2$

$z_0^2 = 3 + 4i$

114.6 (b) A local inverse is any branch of $z = w^{1/2}$ conformal in a neighborhood of $w_0 = z_0^2 = 4$, and giving the value $4^{1/2} = -2$. For instance, $w^{1/2} = \sqrt{\rho}\, e^{i\phi/2}$ where $w = \rho e^{i\phi}$ with $\rho > 0$ and $\pi < \phi < 3\pi$.

114.8 We have $g(f(z)) = z$ near z_0. Differentiating both sides, $g'(f(z))\, f'(z) = 1$, and putting $w = f(z)$, $g'(w)\, f'(z) = 1$, or $g'(w) = \frac{1}{f'(z)}$.

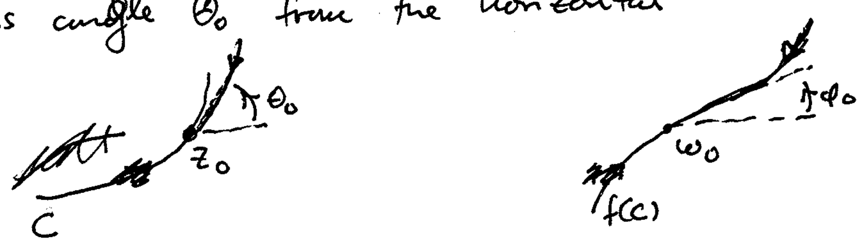
114.10 a) f has Taylor series $f(z) = \sum_1 a_n (z-z_0)^n$ in a neighborhood of z_0, where $a_n = \frac{f^{(n)}(z_0)}{n!}$. The hypotheses give $a_1, \dots, a_{m-1} = 0$, $a_m \neq 0$, and we have $a_0 = f(z_0) = w_0$, so

$$f(z) = w_0 + \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \text{higeher terms}\dots$$

$$f(z) - w_0 = (z-z_0)^m \frac{f^{(m)}(z_0)}{m!}\,(1 + g(z)) \quad \text{where}$$

$g(z) = \frac{1}{a_m}\left(a_{m+1}(z-z_0) + a_{m+2}(z-z_0)^2 + \dots\right)$ is the rest of the Taylor series, analytic at z_0 with $g(z_0) = 0$.

b) If z approaches z_0 along a smooth arc C whose tangent line at z_0 has angle θ_0 from the horizontal



then, for $|z-z_0|$ small, we have $\arg(f(z) - w_0) =$
$\arg\left((z-z_0)^m \frac{f^{(m)}(z_0)}{m!}(1+g(z))\right) = m \arg(z-z_0) + \arg f^{(m)}(z_0) + \arg(1+g(z))$,
and $1 + g(z) \to 1$, so $\arg(1+g(z)) = 0$, $\arg(z-z_0) \to \theta_0$;
$\arg(f(z) - w_0) \to \phi_0$, hence $\phi_0 = m\theta_0 + \arg f^{(m)}(z_0)$.

c) If the angle between curves C_1 and C_2 differs by $\theta_1 - \theta_2 = \alpha$, then the correspond angle between their images differs by $\phi_1 - \phi_2 = m(\theta_1 - \theta_2) = m\alpha$.

115.4 If v is a harmonic conjugate of u, then $f(z) = u + iv$ is analytic, hence so is $-i\, f(z) =$ ~~[illegible]~~ $v - iu$. There~~fore~~ $-u$ is a harmonic conjugate of v. The converse also holds since $-if(z)$ analytic implies $f(z) = i\,(-i\,f(z))$ analytic.

117.2 Composing the harmonic function $h(u,v) = e^{-v}\sin u$ with $w = z^2$, i.e. $u = x^2 - y^2$, $v = 2xy$, where $w = u + iv$, $z = x + iy$, the theorem in §116 implies that $h(x^2-y^2, 2xy) = e^{-2xy}\sin(x^2-y^2)$ is harmonic.

117.4 Since $e^{x+iy} = e^x e^{iy} = e^x\cos y + i e^x \sin y$, the mapping $w = e^z$ is $u = e^x\cos y$, $v = e^x \sin y$. Hence

$$H(x,y) = h(u(x,y), v(x,y)) = 2 - e^x\cos y + \frac{e^x \cos y}{e^{2x}(\cos^2 y + \sin^2 y)} = 2 - e^x\cos y + e^{-x}\cos y.$$

On $x = 0$, this becomes $H(0,y) = 2 - \cos y + \cos y = 2$.

117.5 We have $h_v = -e^{-u}\sin v$, so $h_v(u,0) = 0$ on the u axis.
Composing with $w = z^2$ gives

$$H(x,y) = e^{y^2-x^2}\cos(2xy).$$

On the x-axis, $H_y = 2y\,e^{y^2-x^2}\cos(2xy) - 2x\,e^{y^2-x^2}\sin(2xy)$ gives $H_y(x,0) = 0$ since y and $\sin(2xy)$ vanish.

On the y-axis, $H_x = -2x\,e^{y^2-x^2}\cos(2xy) - 2y\,e^{y^2-x^2}\sin(2xy)$ gives $H_x(0,y) = 0$ since x and $\sin(2xy)$ vanish.

117.6 Adding $2v$ to $h(u,v)$ in the previous problem adds $4xy$ to $H(x,y)$. Thus ~~both~~ the new h and H have $h_v(u,0) = \frac{d}{dv}(2v) = 2$,

$$H_y(x,0) = \frac{\partial}{\partial y}(4xy)\Big|_{y=0} = 4x, \qquad H_x(0,y) = \frac{\partial}{\partial x}(4xy)\Big|_{x=0} = 4y.$$

Thus $h(u,v)$ having constant normal derivative on the u axis (which is the image of the x and y axes) does not imply that $H(x,y)$ has constant normal derivative along the x and y axes.

Add'l Problems

1. $w = iz+1$ maps $0 \mapsto 1$, $1 \mapsto i+1$, $i+1 \mapsto i$, $i \mapsto 0$

2. If $|z|=1$, then $\bar{z} = \frac{1}{z}$, so $w = z + \frac{1}{z} = z + \bar{z} = 2\,\mathrm{Re}(z)$.
Hence each half of $|z|=1$, on which $-1 \le x \le 1$, maps to $[-2,2]$ on the real axis.
If z is real, then so is w. The function $x + \frac{1}{x}$ is decreasing on $(0,1]$ and $[-1,0)$, and increasing on $[1,\infty)$ and $(-\infty,-1]$, so it maps $[1,\infty)$ onto $[2,\infty)$ and $(-\infty,-1]$ onto $(-\infty,2]$.
It also maps $(0,1]$ onto $(2,\infty)$ and $[-1,\infty)$ onto $(-\infty,2]$, and is ~~undefined when~~ at $z=0$, but has $\lim_{z\to 0}(w) = \infty$.

3. As a suitable domain, we'll take $\mathrm{Re}(z) = x > 0$, so $\tan^{-1}(y/x)$ is defined, and $x^2+y^2 \neq 0$, so $\ln(x^2+y^2)$ is also defined.
The simplest approach is to find $f(z)$ directly. If $z = re^{i\theta}$, then $x^2+y^2 = r^2$ and $\tan^{-1}(y/x) = \theta$, so $u(x,y) = 2\ln(r)\,\theta$.
Now $\mathrm{Log}(z) = \ln(r) + i\theta$ (since $-\pi/2 < \theta < \pi/2$), and recalling that $(a+ib)^2 = a^2 - b^2 + 2iab$, we see that

$$2\ln(r)\,\theta = \mathrm{Im}\left(\mathrm{Log}(z)^2\right) = \mathrm{Re}\left(-i\,\mathrm{Log}(z)^2\right).$$

Thus $f(z) = -i\,\mathrm{Log}(z)^2$ has $\mathrm{Re}\, f(z) = u(x,y) = \ln(x^2+y^2)\tan^{-1}(y/x)$
and therefore $v(x,y) = \mathrm{Im}\, f(z) = -\mathrm{Re}\left(\mathrm{Log}(z)^2\right) = \theta^2 - \ln(r)^2 =$

$$\tan^{-1}(y/x)^2 - \frac{1}{4}\ln(x^2+y^2)^2 \quad (+C).$$

This can also be found by integrating u_x with respect to y and using $v_x = -u_y$ to determine the 'constant' of integration $G(x)$.