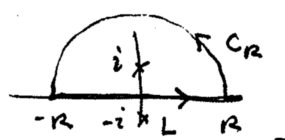
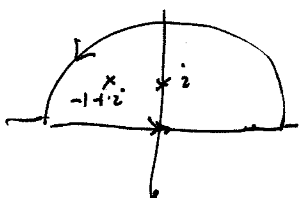
PS7 Solutions

86.2 Evaluate $\int_C \frac{dz}{z^2+1}$ on  . Since $R\left|\frac{1}{z^2+1}\right| \approx \frac{1}{R} \to 0$

as $R \to \infty$, $\int_C \to \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ as $R \to \infty$. By residue theorem,

$\int_C = 2\pi i \operatorname{Res}_{z=i} \frac{1}{z^2+1}$. $\operatorname{Res}_{z=i} \frac{1}{z^2+1} = \operatorname{Res}_{z=i} \frac{1}{(z-i)(z+i)} = \frac{1}{i+i} = \frac{1}{2i}$,

so $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \frac{2\pi i}{2i} = \pi$, $\int_0^{\infty} \frac{dx}{x^2+1} = \pi/2$. [In this case we can also easily find the indefinite integral $\int \frac{dx}{1+x^2} = \tan^{-1}x + C$].

86.8 The norm of the integrand is $\approx \frac{1}{R^3}$ on $|z|=R$, so same contour as in 86.2 works. The roots of $z^2+2z+2=0$ are $z=-1\pm i$, so we have 

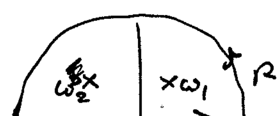
$$\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)} = 2\pi i\left(\operatorname{Res}_{z=i} f(z) + \operatorname{Res}_{z=i-1} f(z)\right)$$

where $f(z) = \frac{z}{(z^2+1)(z^2+2z+2)}$.

$$\operatorname{Res}_{z=i} f(z) = \operatorname{Res}_{z=i} \frac{z}{(z-i)(z+i)(z^2+2z+2)} = \frac{i}{2i(2i+1)} = \frac{1-i2}{10}$$

$$\operatorname{Res}_{z=i-1} f(z) = \operatorname{Res}_{z=i-1} \frac{z}{(z^2+1)(z+1-i)(z+1+i)} = \frac{i-1}{(1-2i)(2i)} = \frac{-1+i3}{10}$$

$$2\pi i\left(\operatorname{Res}_{z=i} f(z) + \operatorname{Res}_{z=i-1} f(z)\right) = 2\pi i \cdot \frac{i}{10} = \boxed{-\pi/5}$$

88.4 Since $\left|\frac{z e^{iaz}}{z^4+4}\right| \lesssim R^3$ for $|z|=R$, $\operatorname{Im} z>0$ with $a>0$, we can evaluate on contour  to get

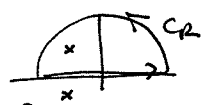
$$\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\,dx = \operatorname{Im}\int_{-\infty}^{\infty} \frac{x\,e^{iax}}{x^4+4}\,dx = \int_C \frac{z\,e^{iaz}}{z^4+4}\,dz = 2\pi i\left(\operatorname{Res}_{z=\omega} + \operatorname{Res}_{z=\omega^3}\right)\frac{z\,e^{iaz}}{z^4+4},$$

where $\omega_1 = i+1$, $\omega_2 = i-1$ are 4th root of -4. (Jordan's Lemma is not needed).

$$\operatorname{Res}_{z=\omega_1} \frac{z e^{iaz}}{z^4+4} = \frac{\omega e^{ia\omega}}{4\omega^3} = \frac{-i}{8} e^{ia\omega} = \frac{-i}{8} e^{ia} e^{-a}$$

$$\operatorname{Res}_{z=\omega_2} \frac{z e^{iaz}}{z^4+4} = \frac{\omega_2 e^{ia\omega_2}}{4\omega_2^3} = \frac{i}{8} e^{ia\omega_2} = \frac{i}{8} e^{-ia} e^{-a}$$

$$\operatorname{Res}_{z=\omega_1} + \operatorname{Res}_{z=\omega_2} = \frac{-i}{8} e^{-a}(e^{ia}-e^{-ia}) = \frac{1}{4} e^{-a}\sin a, \text{ so } \int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4} = \frac{\pi}{2} e^{-a} \sin a$$

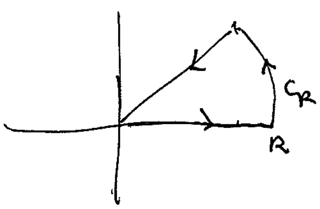
88.9 $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+2x+2}\,dx = \operatorname{Im} \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+2x+2}\,dx$. Since $\left| \frac{z}{z^2+2z+2} \right| \approx \frac{1}{R} \to 0$ on C_R, Jordan's Lemma allows us to evaluate this using a contour integral $\int \frac{z e^{iz}}{z^2+2z+2}\,dz$ on .

The roots of $z^2+2z+2 = (z+1)^2+1 = 0$ are $z = -1 \pm i$, so our contour integral is given by $2\pi i \operatorname{Res}_{z=i-1} \frac{z e^{iz}}{(z+1-i)(z+1+i)} = 2\pi i \frac{(i-1)e^{-1-i}}{2i}$

$= \pi (i-1) e^{-1} (\cos 1 - i \sin 1)$.

Taking the imaginary part gives $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+2x+2}\,dx = \pi e^{-1}(\cos(1) + \sin(1))$.

88.12 (a) e^{iz^2} is entire, so $\int_C e^{iz^2}\,dz = 0$. On the contour



this gives $\int_0^R e^{ix^2}\,dx + \int_R^0 e^{i(\omega r)^2}\,\omega\,dr + \int_{C_R} e^{iz^2}\,dz = 0$,

where $\omega = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$ and $\omega^2 = -1$.

Rewriting this as $\int_0^R e^{ix^2}\,dx = \frac{1+i}{\sqrt{2}} \int_0^R e^{-r^2}\,dr - \int_{C_R} e^{iz^2}\,dz$

gives $\int_0^R \cos(x^2)\,dx = \frac{1}{\sqrt{2}} \int_0^R e^{-r^2}\,dr - \operatorname{Re} \int_{C_R} e^{iz^2}\,dz$

$\int_0^R \sin(x^2)\,dx = \frac{1}{\sqrt{2}} \int_0^R e^{-r^2}\,dr - \operatorname{Im} \int_{C_R} e^{iz^2}\,dz$

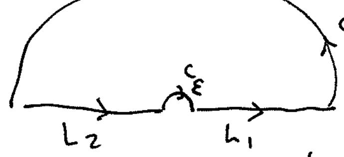
(b) Parametrizing C_R as $z = Re^{i\theta}$ for $0 \le \theta \le \pi/4$, we get $e^{iz^2} = e^{iR^2(\cos(2\theta) + i\sin(2\theta))} = e^{-R^2 \sin(2\theta)} e^{iR^2 \cos(2\theta)}$, so $|e^{iz^2}| = e^{-R^2 \sin(2\theta)}$. Letting $\theta = \varphi/2$, we get

$$\left| \int_{C_R} e^{iz^2}\,dz \right| \le \int_0^{\pi/2} e^{-R^2 \sin(\varphi)} R \left| \tfrac{i}{2} e^{i\varphi} \right| d\varphi = \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin(\varphi)}\,d\varphi \le \frac{R}{2} \frac{\pi}{2R^2} = \pi/4R.$$

(c) As $R \to \infty$, $\int_{C_R} e^{iz^2}\,dz \to 0$ and we are left with

$$\int_0^{\infty} \cos(x^2)\,dx = \int_0^{\infty} \sin(x^2)\,dx = \frac{1}{\sqrt{2}} \int_0^{\infty} e^{-r^2}\,dr = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

91.1 Let $f(z) = (e^{iaz} - e^{ibz})/z^2$. The Taylor series of $e^{iaz} - e^{ibz}$ is $(1 + iaz + \dots) - (1 + ibz + \dots) = i(a-b)z + \dots$, so f has a simple pole with residue $i(a-b)$ at $z=0$, and is analytic everywhere else.

On 

, we have $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$

for $a, b \geq 0$ since $|f(z)| \leq \frac{2}{R^2}$, and $\int_{C_\varepsilon} f(z)\,dz \to \pi(a-b)$

$= -\frac{1}{2}(2\pi i \,\mathrm{Res}_{z=0} f(z))$ by Theorem in § 89. By Cauchy-Goursat, the whole $\int_C f(z)\,dz$ is zero, hence

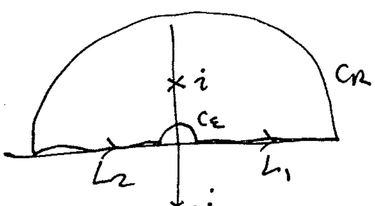
$$\lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \left(\frac{e^{iaz} - e^{ibz}}{z^2} \right) dz = \pi(b-a)$$

The real part of the left hand side is $\int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2}\,dx$

(note that $\cos(ax) - \cos(bx) = (1 - a^2x^2 + \dots) - (1 - b^2x^2 + \dots) = (b^2 - a^2)x^2 + \dots$, so $\frac{\cos(ax) - \cos(bx)}{x^2}$ has a limit as $x \to 0$, and this integral is not improper at $x = 0$). The integrand is even, so dividing by 2 gives $\int_0^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a)$.

In particular, taking $a = 0$ and $b = 2$ and using $1 - \cos(2x) = 2\sin^2 x$, we get $2\int_0^{\infty} \frac{\sin^2(x)}{x^2}\,dx = \frac{\pi}{2}(2-0)$; $\int_0^{\infty} \frac{\sin^2(x)}{x^2}\,dx = \pi/2$.

91.2 Using a branch of $z^{-1/2} = e^{-\frac12 \log z}$ with $\mathrm{Im}(\log z) = \arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$, it is analytic on $\mathbb{C} - \{yi \mid y \leq 0\}$. Integrating $f(z) = \frac{z^{-1/2}}{z^2+1}$ on



we have $\varepsilon|f(z)| \approx \varepsilon \cdot \varepsilon^{-1/2} = \varepsilon^{1/2} \to 0$ as $\varepsilon \to 0$ on C_ε

$R|f(z)| \approx R\frac{R^{-1/2}}{R^2} \approx R^{-3/2} \to 0$ as $R \to \infty$ on C_R.

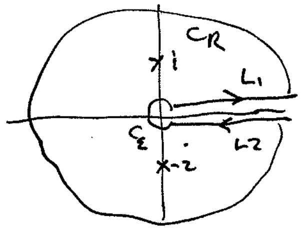
Therefore $\int_{C_R} f(z)$ and $\int_{C_\varepsilon} f(z) \to 0$. ~~leaving~~ On the negative real axis, $z^{-1/2} = \frac{1}{i\sqrt{x}}$ so $\int_{L_2} f(z)\,dz = -i\int_{\infty}^{0} \frac{-dx}{\sqrt{x}(x^2+1)} = -i\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2+1)}$

On the positive real axis, $\int_{L_1} f(z)\,dz = \int_0^{\infty} \frac{dx}{\sqrt{x}(x^2+1)}$.

Hence $(1-i)\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2+1)} = \int_C f(z)\,dz = 2\pi i\,\mathrm{Res}_{z=i} \frac{z^{-1/2}}{(z-i)(z+i)} = 2\pi i \frac{e^{-i\pi/4}}{2i} = \pi\frac{1-i}{\sqrt{2}}$,

so $\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$.

91.3 To do the previous integral using the contour



we note that $\int_{C_R} f(z)\,dz$, $\int_{C_\varepsilon} f(z)\,dz \to 0$ just as before.

Also $\int_{L_1} \frac{z^{-1/2}}{z^2+1}\,dz = \int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)}$, as before, but on the lower side of the branch cut, $z^{-1/2} = e^{-\frac{1}{2}(2\pi i)} \cdot x^{-1/2} = \frac{-1}{\sqrt{x}}$.

Thus $\int_{L_2} \frac{z^{-1/2}}{z^2+1}\,dz = -\int_\infty^0 \frac{dx}{\sqrt{x}\,(x^2+1)} = \int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)}$, giving

$$2\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)} = \int_c \frac{z^{-1/2}\,dz}{z^2+1} = 2\pi i\left(\mathrm{Res}_{z=i} + \mathrm{Res}_{z=-i}\right)\left(\frac{z^{-1/2}}{(z-i)(z+i)}\right).$$

On this branch, $i^{-1/2} = e^{-i\pi/4} = \frac{1-i}{\sqrt{2}}$, $(-i)^{1/2} = e^{-i\,3\pi/4} = \frac{-1-i}{\sqrt{2}}$,

so $\mathrm{Res}_{z=i} = \frac{1-i}{\sqrt{2}\,(2i)} = \frac{1-i}{2\sqrt{2}}$ $\quad \mathrm{Res}_{z=-i} = \frac{-1-i}{\sqrt{2}(-2i)} = \frac{-1-i}{2\sqrt{2}}$,

$2\pi i\left(\mathrm{Res}_{z=i} + \mathrm{Res}_{z=-i}\right) = 2\,\pi/\sqrt{2}$, giving $\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)} = \frac{\pi}{\sqrt{2}}$

91.5 setting $t = \frac{1}{x+1}$ in $\int_0^1 t^{p-1}(1-t)^{q-1}\,dt$ gives (since $1-t = \frac{x}{x+1}$ and $x = \frac{1-t}{t}$)

$$B(p,q) = \int_\infty^0 \frac{1}{(x+1)^{p-1}}\,\frac{x^{q-1}}{(x+1)^{q-1}}\,\frac{-dx}{(x+1)^2} = \int_0^\infty \frac{x^{q-1}}{(x+1)^{p+q}}\,dx.$$

If $p+q=1$, this becomes

$$B(p,1-p) = \int_0^\infty \frac{x^{-p}}{x+1}\,dx = \frac{\pi}{\sin \pi p}$$ by §91 (5).

~~Since $\sin(\pi(1-q)) = \sin(\pi-\pi q) = -\sin(-\pi q) = \sin \pi q$,~~

92.5 Since $\frac{1}{(a+\cos\theta)^2}$ is even, $\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2}$.

Put $z = e^{i\theta}$, $dz = ie^{i\theta}\,d\theta = iz\,d\theta$, $d\theta = \frac{dz}{iz}$ to get

$$\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \oint_{|z|=1} \frac{dz}{iz\,(a+(z+z^{-1})/2)^2} = \frac{4}{i}\oint_{|z|=1} \frac{dz}{z\,(z+2a+z^{-1})^2} = \frac{4}{i}\oint \frac{z\,dz}{(z^2+2az+1)^2}.$$

For $a>1$, $z^2+2az+1=0$ has real roots $\frac{-2a \pm \sqrt{4a^2-4}}{2} = -a \pm \sqrt{a^2-1}$. One of them, $-a+\sqrt{a^2-1}$, is inside the unit circle (since their product is 1 and the other root $-a-\sqrt{a^2-1}$ is <-1). Thus

$$\frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = 4\pi\,\mathrm{Res}_{z=-a+\sqrt{a^2+1}}\,\frac{z}{(z^2+2az+1)^2}$$

To find the residue, let $\alpha = -a + \sqrt{a^2-1}$, $\beta = -a - \sqrt{a^2-1}$.

Then we need to expand $\frac{z}{(z-\alpha)^2(z-\beta)^2}$ in a Laurent series at $z=\alpha$,

so let $w = z-\alpha$, so $z = w+\alpha$, $z-\beta = w + \alpha - \beta = w + 2\sqrt{a^2-1}$.

This gives $\frac{1}{w^2}\frac{w+\alpha}{(w+(\alpha-\beta))^2}$, and the power series expansion of

$$\frac{\alpha+w}{(\alpha-\beta+w)^2} \text{ is } (\alpha+w)(\alpha-\beta+w)^{-2} = (\alpha+w)(\alpha-\beta)^{-2}\left(1+\frac{w}{\alpha-\beta}\right)^{-2}$$

$$= \frac{1}{(\alpha-\beta)^2}(\alpha+w)\left(1 - 2\frac{w}{\alpha-\beta} + O(w^2)\right)$$

$$= \frac{1}{(\alpha-\beta)^2}\left(\alpha + \left(1 - \frac{2\alpha}{\alpha-\beta}\right)w + O(w^2)\right)$$

$$= \frac{1}{(\alpha-\beta)^2}\left(\alpha - \frac{\alpha+\beta}{\alpha-\beta}w + O(w^2)\right)$$

Thus $\frac{1}{w^2}\frac{w+\alpha}{(w+(\alpha-\beta))^2} = \frac{1}{(\alpha-\beta)^2}\left(\frac{\alpha}{w^2} - \frac{\alpha+\beta}{\alpha-\beta}w^{-1} + \cdots\right)$

Giving residue $\frac{-(\alpha+\beta)}{(\alpha-\beta)^3} = \frac{2a}{8(\sqrt{a^2-1})^3} = \frac{a}{4(\sqrt{a^2-1})^3}$.

Then $\int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2} = \frac{4\pi a}{4(\sqrt{a^2-1})^3} = \frac{a\pi}{(\sqrt{a^2-1})^3}$ $(a>1)$.

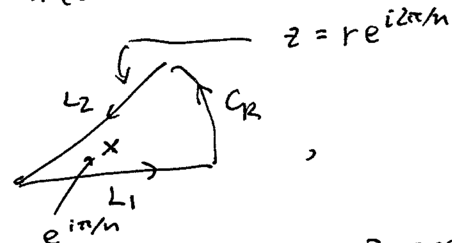
94.5 As in the proof of the argument principle, $\frac{f'(z)}{f(z)}$ has a ~~pole~~ simple pole with residue m_k at each zero z_k of $f(z)$. Then $\frac{zf'(z)}{f(z)}$ has a simple pole pole with residue $m_k z_k$ at z_k, unless $z_k = 0$. But if $z_k = 0$, although there is no longer a pole at z_k, we also have $m_k z_k = 0$, so the $m_k z_k$ term doesn't contribute to the sum. The residue theorem now gives $\int_C \frac{zf'(z)}{f(z)}dz = 2\pi i\sum_k m_k z_k$.

94.8 On $|z|=2$, $|2z^5| = 2^6 = 64$ and $|6z^2+z+1| \le 24+2+1 = 27$. Hence all 5 roots are inside $|z|<2$. On $|z|=1$, $|6z^2| = 6$ and $|2z^5+z+1| \le 2+1+1 = 4$. By Rouche's Theorem, $2z^5-6z^2+z+1=0$ has the same number of roots in $|z|<1$ as $6z^2$ does, namely 2. So there remain 3 roots in the annulus $1 \le |z| < 2$.

Additional Problems

1. a) The roots of $z^n+1=0$ are the n^{th} roots of -1; they are all distinct, so $\frac{1}{z^n+1}$ has a simple pole at each of them, in particular at $z=e^{i\pi/n}$.
The residue is then $\frac{1}{f'(e^{i\pi/n})} = \frac{1}{n(e^{i\pi/n})^{n-1}} = \frac{e^{-i\pi(n-1)/n}}{n} = \frac{e^{-i\pi}e^{i\pi/n}}{n} = \frac{-e^{i\pi/n}}{n}$.

b) Evaluating $\int_C \frac{dz}{z^n+1}$ on



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we have $R\left|\frac{1}{z^n+1}\right| \approx \frac{R}{R^n} = \frac{1}{R^{n-1}} \to 0$ as $R\to\infty$ on C_R, so $\int_{C_R} \to 0$. On L_2, $\int_{L_2} \frac{dz}{z^n+1} = \int_\infty^0 \frac{e^{i2\pi/n}\,dr}{r^n+1}$ because $(re^{i2\pi/n})^n = r^n$.
On L_1 we get $\int_0^\infty \frac{dx}{x^n+1}$. The only n^{th} root of -1 in this sector is $e^{i\pi/n}$, so $\int_C \frac{dz}{z^n+1} = 2\pi i \operatorname{Res}_{z=e^{i\pi/n}} \frac{1}{z^n+1} = \frac{-2\pi i\, e^{i\pi/n}}{n}$ by part (a).

This gives $\left(1-e^{i2\pi/n}\right)\int_0^\infty \frac{dx}{x^n+1} = \frac{-2\pi i\, e^{i\pi/n}}{n}$.

Dividing by $-2ie^{i\pi/n}$, we get $\sin\frac{\pi}{n}\int_0^\infty \frac{dx}{x^n+1} = \frac{\pi}{n}$, $\int_0^\infty \frac{dx}{x^n+1} = \frac{\pi}{n\sin(\pi/n)}$.

2. a) $f(x) = x - e^{-x}$ is monotone increasing, continuous, and $f(0) = -1 < 0$, $f(1) = 1-\frac{1}{e} > 0$, so there is a solution x_0 of $x - e^{-x} = 0$ in the interval $(0,1)$, and it is the only real solution.

b) On $Re(z) \geq x_0$ we have $|z| \geq Re(z) \geq x_0$, $|e^{-z}| = e^{-Re(z)} \leq e^{-x_0} = x_0$. If $z \neq x_0$, ~~both inequalities are~~ the first inequality is strict, i.e. $|z| > x_0$, so $|z| > |e^{-z}|$ on any contour in $Re(z) \geq x_0$ that does not pass through x_0. ~~By~~ If $z = e^{-z}$ had another solution in $Re(z) \geq x_0$, it would be inside some such contour C. But this contradicts Rouché's theorem, which implies that $z - e^{-z}$ has the same number zeroes inside C as z, namely, none. (because $z=0$ is not inside C, since it's not in $Re(z) \geq x_0$, as $x_0 > 0$).