PS 6 Solutions

$\boxed{29.5}$ We want to prove that $\overline{f(\bar z)} = -f(z)$. To do this we first show that $\overline{f(\bar z)}$ is analytic. Let $f(z) = u(x,y) + i v(x,y)$. Then
$\overline{f(\bar z)} = U(x,y) + i V(x,y)$, where $U(x,y) = u(x,-y)$ and $V(x,y) = -v(x,-y)$, and we can check the Cauchy-Riemann equations. This part is the same as in the proof of the Reflection Principle.

Now, if $z = x$ is real, then $\overline{f(\bar z)} = \overline{f(x)} = -f(x)$ because we assume $f(x)$ pure imaginary. So $F(z) = \overline{f(\bar z)}$ and $-f(z)$ are analytic and agree on the intersection of D with the real axis. Then they agree on all of D, i.e. $\overline{f(z)} = -f(\bar z)$, by the uniqueness theorem.

$\boxed{65.4}$ Let $z = \pi/2 + \omega$, so $\omega = z - \pi/2$.

Then $\cos(z) = -\sin(z - \pi/2) = -\sin(\omega) = -\omega + \frac{\omega^3}{3!} - \frac{\omega^5}{5!} + \ldots = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\omega^{2n+1}}{(2n+1)!}$.

Substitute $\omega = z - \pi/2$ to get
$$\cos(z) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(z-\pi/2)^{2n+1}}{(2n+1)!},$$
valid for all z since $\cos(z)$ is entire.

$\boxed{65.8}$ (a) Using $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$ gives
$$\cos z = \frac{1}{2}\sum_{n=0}^{\infty} i^n \frac{z^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n i^n \frac{z^n}{n!}.$$
For n odd, the terms $\frac{1}{2} i^n \frac{z^n}{n!}$ and $\frac{1}{2}(-1)^n i^n \frac{z^n}{n!}$ cancel. For n even they are equal and add up to $i^n \frac{z^n}{n!}$. Putting $n = 2m$, $i^n = (-1)^m$, giving
$$\cos z = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!}$$

(b) The derivatives of $\cos z$ are

n	0	1	2	3	4	....
$(\frac{d}{dz})^n \cos(z)$	$\cos z$	$-\sin z$	$-\cos z$	$\sin z$	$\cos z$	repeats...

Evaluating at $z = 0$, we get 0 for n odd and $(-1)^m$ for $n = 2m$ even. Then Taylor's formula gives
$$\cos z = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!}$$

$\boxed{68.1}$ Multiply $\sin\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(1/z^2)^{2n+1}}{(2n+1)!} = \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!\, z^{4n}}$

by z^2 to get

$$z^2 \sin\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{4n}} = 1 - \frac{1}{3!} z^{-4} + \frac{1}{5!} z^{-8} - \cdots$$

$\boxed{68.3}$ To get a series expansion for $\frac{1}{z(1+z^2)} = \frac{1}{z} \frac{1}{1+z^2}$, we expand the geometric series in powers of z^{-2}:

$$\frac{1}{z} \frac{1}{1+z^2} = \frac{1}{z} \frac{z^{-2}}{1+z^{-2}} = z^{-3}\left(1 - z^{-2} + z^{-4} - \cdots\right)$$

$$= z^{-3} - z^{-5} + z^{-7} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{-2n-3} \quad \left(\text{or, } = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{2n+1}}\right).$$

This is valid for $|z^{-2}| < 1$, thus for $|z| > 1$.

$\boxed{68.5}$ On $|z|<1$, expand both geometric series in powers of z:

$$\frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2} \frac{1}{1-z/2}$$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - 1\right) z^n = -\frac{1}{2} - \frac{3}{4} z - \frac{7}{8} z^2 - \cdots$$

This is valid since $|z|<1$ and $|z/2|<1$ for $|z|<1$.

On $1<|z|<2$ we still have $|z/2|<1$, and now $\left|\frac{1}{z}\right|<1$, so we expand the first term in powers of z^{-1}:

$$\frac{1}{z-1} - \frac{1}{z-2} = z^{-1} \frac{1}{1-z^{-1}} + \frac{1}{2} \frac{1}{1-z/2} = \sum_{n=1}^{\infty} z^{-n} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n}$$

$$= \sum_{n=-\infty}^{\infty} a_n z^n, \quad \text{where } a_n = \begin{cases} \frac{1}{2^{n+1}} & n \geq 0 \\ 1 & n<0 \end{cases}$$

$$= \cdots + z^{-2} + z^{-1} + \frac{1}{2} + \frac{1}{4} z + \frac{1}{8} z^2 + \cdots$$

On $|z|>2$, expand both terms in negative powers:

$$\frac{1}{z-1} - \frac{1}{z-2} = z^{-1} \frac{1}{1-z^{-1}} - z^{-1} \frac{1}{1-2/z} = \sum_{n=1}^{\infty} \frac{1}{z^n} - \sum_{n=1}^{\infty} \frac{2^{n-1}}{z^n}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} = -z^{-2} - 3z^{-3} - 7z^{-4} - \cdots$$

68.9 a) The formula for the coefficients of a Laurent series gives

$$e^{z(w-w^{-1})/2} = \sum_{n=-\infty}^{\infty} J_n(z)\, w^n, \quad \text{where}$$

$$J_n(z) = \frac{1}{2\pi i}\int_C \frac{e^{z(w-w^{-1})/2}}{w^{n+1}}\, dw, \quad \text{valid for } 0<|w|<\infty, \text{ since}$$

$e^{z(w-w^{-1})/2}$ is an analytic function of w on this domain, where C is a positively oriented closed contour enclosing 0. Taking C to be $w = e^{i\phi}$ on $-\pi \le \phi \le \pi$, this gives

$$J_n(z) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{-(n+1)i\phi}\, e^{z(i\sin\phi)}\, i e^{i\phi}\, d\phi,$$

since $(w-w^{-1})/2 = (e^{i\phi}-e^{-i\phi})/2 = i\sin\phi$. This simplifies to

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\sin\phi - ni\phi}\, d\phi .$$

b) $e^{i(\sin\phi - n\phi)} = \cos(\sin\phi - n\phi) + i\sin(\sin\phi - n\phi)$.

Since $\sin(\sin\phi - n\phi)$ is an odd function, the integrals on $\int_{-\pi}^{0}$ and $\int_{0}^{\pi}$ cancel on this term. Since $\cos(\sin\phi - n\phi) = \cos(n\phi - \sin\phi)$ is an even function, the whole integral is twice the integral on $\int_0^{\pi}$, giving

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - \sin\phi)\, d\phi .$$

72.1 Differentiating $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+\dots$

term by term gives

$$-(-(1-z)^{-2}) = \frac{1}{(1-z)^2} = \underbrace{\sum_{n=0}^{\infty} nz^{n-1}}_{(n=0 \text{ term is } 0)} = \underbrace{\sum_{n=1}^{\infty} nz^{n-1}}_{(\text{shift index to } n-1)} = \sum_{n=0}^{\infty}(n+1)z^n = 1+2z+3z^2+\dots$$

Same procedure again gives

$$-(-2(1-z)^{-3}) = \frac{2}{(1-z)^3} = \sum_{n=0}^{\infty}(n+1)nz^{n-1} = \sum_{n=1}^{\infty}(n+2)(n+1)z^n = 2+3\cdot 2z+4\cdot 3z^2+\dots$$

All valid on $|z|<1$.

~~72.2 Let $w=(1-z)^{-1}$ in the above to get~~

~~$w^3 = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{(w-1)^n}$~~

$\boxed{72.2}$ Note that $\dfrac{1}{1-1/(1-z)} = \dfrac{1-z}{1-z-1} = \dfrac{1-z}{-z} = 1 - \dfrac{1}{z}$.

Substituting $\frac{1}{1-z}$ for z in $\left(\frac{1}{1-z}\right)^2 = \sum_{n=0} (n+1) z^n$ therefore gives

$$\left(1-\frac{1}{z}\right)^2 = \frac{1}{z^2}(z-1)^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{(z-1)^n}.$$

Divide by $(z-1)^2$:

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{(z-1)^{n+2}} \underset{\substack{\text{shift index} \\ \text{to } n+2}}{=} \sum_{n=2}^{\infty} \frac{(-1)^n (n-1)}{(z-1)^n} = \frac{1}{(z-1)^2} - \frac{2}{(z-1)^3} + \frac{3}{(z-1)^4} - \cdots$$

The series we started with converged for $|z|<1$, so the last series converges for $\left|\frac{1}{1-z}\right| < 1$, i.e. $|z-1| > 1$.

$\boxed{72.3}$ $\dfrac{1}{z} = \dfrac{1}{2}\,\dfrac{1}{1+(z-2)/2} = \dfrac{1}{2}\sum_{n=0}^{\infty} (-1)^n \dfrac{(z-2)^n}{2^n}$ $\quad (|z-2|<2)$

$$\Rightarrow -\frac{1}{z^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}(z-2)^{n-1} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+1)}{2^{n+1}}(z-2)^n$$

Now change signs and write $2^{n+1} = 2^n \cdot 2$ to get

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^n}(z-2)^n \qquad \text{for } |z-2|<2$$

$\boxed{72.4}$ On $0<|z|<\infty$, we have the Laurent series

$$\begin{aligned}
\frac{1-\cos z}{z^2} &= \frac{1}{z^2}\left(1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)\right) \\
&= \frac{1}{z^2}\left(\frac{z^2}{2!}-\frac{z^4}{4!}+\cdots\right) \\
&= \frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\cdots = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n+2)!}
\end{aligned}$$

It has no negative z powers, so it's the Taylor series of an analytic function $f(z)$ with $f(0) = \frac{1}{2!} = \frac{1}{2}$. The series has radius of convergence ∞, so the resulting function

$$f(z) = \begin{cases} (1-\cos z)/z^2 & z \neq 0 \\ 1/2 & z = 0 \end{cases}$$

is entire.

$\boxed{72.6}$ On $|z-1|<1$ [and more generally on $\mathbb{C}-(\mathbb{R}_{\le 0})$], $\operatorname{Log}(z)$ is the antiderivative of $\frac{1}{z}$ with value $\operatorname{Log}(1)=0$ at $z=1$. On $|z-1|<1$ we can antidifferentiate the Taylor series

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

term by term to get

$$\operatorname{Log}(z) = C + \sum_{n=0}^{\infty} \frac{(-1)^n (z-1)^{n+1}}{n+1} = C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n$$

on $|z-1|<1$, and use $\operatorname{Log}(1)=0$ to get the constant of integration $C=0$, $\operatorname{Log}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n$.

$\boxed{72.7}$ Divide the previous series by $z-1$ to get

$$\frac{\operatorname{Log}(z)}{z-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (z-1)^n,$$

which is then the Taylor series about $z=1$ of an analytic function

$$f(z) = \begin{cases} \frac{\operatorname{Log}(z)}{z-1} & z \neq 1 \\ 1 & z=1 \end{cases}$$

[since the constant term of the series is $\frac{(-1)^0}{0+1} = 1$.].

This shows $f(z)$ is analytic on $|z-1|<1$, but the ~~expression~~ function $\frac{\operatorname{Log}(z)}{z-1}$ is also analytic for all $z \neq 0, 1$ with $-\pi < \operatorname{Arg} z < \pi$, so $f(z)$ is analytic for all $z \neq 0$, $-\pi < \operatorname{Arg} z < \pi$.

$\boxed{72.11}$ For $|z|<1$ we have the convergent geometric series

$$\sum_{n=0}^{\infty} (-1)^n z^{2n} = \sum_{n=0}^{\infty} (-z^2)^n = \frac{1}{1-(-z^2)} = \frac{1}{1+z^2};$$

then $\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$ extends this to an analytic function for all $z \neq \pm i$.

$\boxed{73.4}$ $$\frac{1}{e^z-1} = \left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots\right)^{-1}$$

$$= z^{-1}\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \dots\right)^{-1}$$

Finding $\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \dots\right)^{-1}$ term by term ($O(z^k)$ stands for terms $a_k z^k + \dots$)

$$\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \dots\right)(A + O(z)) = A + O(z) \Rightarrow A = 1$$

$$\left(1 + \frac{z}{2} + \frac{z^2}{6} + z^3/24 + \dots\right)(1 + Bz + O(z^2)) = 1 + Bz + \frac{z}{2} + O(z^2) \Rightarrow B = -\frac{1}{2}$$

$$\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \dots\right)\left(1 - \frac{1}{2}z + Cz^2 + O(z^3)\right) = 1 + 0z + \frac{z^2}{6} - \frac{z^2}{4} + Cz^2 + O(z^3)$$

$$\Rightarrow C = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

$$\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \dots\right)\left(1 - \frac{1}{2}z + \frac{1}{12}z^2 + Dz^3 + \dots\right)$$

$$= 1 + 0z + 0z^2 + \frac{z^3}{24} - \frac{z^3}{12} + \frac{z^3}{12} + Dz^3 + O(z^4)$$

$$\Rightarrow D = 0$$

$$\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \dots\right)\left(1 - \frac{1}{2}z + \frac{1}{12}z^2 + 0z^3 + Ez^4 + \dots\right)$$

$$= 1 + 0z + 0z^2 + 0z^3 + \frac{z^4}{120} - \frac{z^4}{48} + \frac{z^4}{72} + Ez^4 + O(z^5)$$

$$\frac{1}{120} - \frac{1}{48} + \frac{1}{72} = \frac{6 - 15 + 10}{720} = \frac{1}{720} \Rightarrow E = \frac{-1}{720}.$$

So $\frac{1}{e^z-1} = z^{-1}\left(1 - \frac{1}{2}z + \frac{1}{12}z^2 - \frac{1}{720}z^4 + \dots\right) = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \dots$

The poles of $\frac{1}{e^z-1}$ occur where $e^z = 1$, i.e. for $z = 2\pi i n$, and the function is analytic elsewhere. Hence its Laurent series about the pole at $z=0$ converges on $0 < |z| < 2\pi$, since the poles are outside this annulus.

$\boxed{77.1}$ (e) $\sinh z = z + z^3/6 + \dots \qquad \frac{1}{1-z^2} = 1 + z^2 + z^4 + \dots$

$$\frac{\sinh(z)}{1-z^2} = (z + z^3/6 + \dots)(1 + z^2 + \dots) = \left(z + z^2 + \frac{7z^3}{6} + \frac{7z^4}{6} + \dots\right)$$

$$\frac{\sinh(z)}{z^4(1-z^2)} = z^{-3} + z^{-2} + \boxed{\frac{7}{6}}z^{-1} + \frac{z}{6} + \dots$$

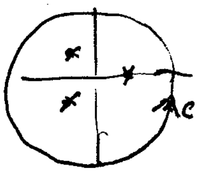
$$= \operatorname{Res}_{z=0} \frac{\sinh(z)}{z^4(1-z^2)}$$

$\boxed{77.2}$ (c) $z^2 e^{1/z}$ has only singularity, at $z=0$, with Laurent series

$$z^2\left(1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\frac{1}{3!}\frac{1}{z^3}+\dots\right)$$

and residue $\frac{1}{3!}=\frac{1}{6}$. By residue theorem, $\int_C z^2 e^{1/z}\,dz = \frac{2\pi i}{3!} = \frac{\pi i}{3}$ for the circle $|z|=3$ or any positively oriented simple close contour enclosing $z=0$.

$\boxed{77.4}$ (a) The three poles, at 1, $e^{\pm 2\pi i/3}$ are all inside the contour $|z|=2$,



so we can evaluate the integral using the residue at ∞, that is by making the change of variables $z=1/w$ to get

$$\int_{|w|=1/2} \frac{-w^{-5}}{1-w^{-3}}\, w^{-2}\,dw = \int_{|w|=1/2} \frac{w^{-7}}{1-w^{-3}}\,dw = 2\pi i \operatorname{Res}_{w=0} \frac{w^{-7}}{1-w^{-3}}.$$

Now $\frac{w^{-7}}{1-w^{-3}} = \frac{-w^{-4}}{1-w^{3}} = -w^{-4}(1+w^3+\dots) = -w^{-4} - w^{-1} + \dots$ has residue -1 at $w=0$, giving $\int_C \frac{z^5}{1-z^3}\,dz = -2\pi i$ (in other words, $\frac{z^5}{1-z^3}$ has residue 1 at ∞).

$\boxed{77.7}$ Let $f(z)=\frac{P(z)}{Q(z)}$. Then $z^{-2}f(z^{-1}) = \frac{z^{-2}P(z^{-1})}{Q(z^{-1})}$. If P, Q are polynomials as specified, this is

$$\frac{a_n z^{-n-2} + a_{n-1} z^{-n-1} + \dots + a_0 z^{-2}}{b_m z^{-m} + b_{m-1} z^{-m+1} + \dots + b_0}$$

$$= z^{m-n-2}\,\frac{a_n + a_{n-1} z + \dots + a_0 z^n}{b_m + b_{m-1} z + \dots + b_0 z^m}.$$

Since $b_m \neq 0$, the last factor is analytic at $z=0$, and if $m \geq n+2$, then so is z^{m-n-2}. So $\frac{1}{z^2} f(\frac{1}{z})$ has a removable singularity and residue 0 at $z=0$, giving $\int_C \frac{P(z)}{Q(z)}\,dz = 0$ for any contour C enclosing all the poles (i.e., all the zeroes of $Q(z)$).

[This can also be shown taking C to be the circle $|z|=R$ and observing that $m \geq n+2$ implies $\lim_{R\to\infty} R\cdot M_R = 0$, where M_R is the maximum of $\left|\frac{P(z)}{Q(z)}\right|$ on $|z|=R$.]

$\boxed{79.1}$ (a) singular point $z=0$, principal part $\frac{1}{2!}z^{-1}+\frac{1}{3!}z^{-2}+\dots$, essential.

(b) Singular point $z=-1$, principal part the negative power terms in

$$\frac{z^2}{z+1}=\frac{((z+1)-1)^2}{z+1}=(z+1)^{-1}-2+(z+1)^1,$$

that is, $(z+1)^{-1}$, pole of order 1.

(c) Singular point $z=0$, princ. part 0 since z divides the Taylor series of $\sin z$, removable (by extending to $f(z)=\begin{cases}(\sin z)/z & z\neq 0\\ 1 & z=0\end{cases}$

(d) Singular point $z=0$, princ. part z^{-1} (the first term of $\frac{\cos z}{z}=z^{-1}-\frac{1}{2!}z+\dots$), pole of order 1.

(e) Singular point $z=2$, princ. part $\frac{-1}{(z-2)^3}$, pole of order 3.

$\boxed{79.3}$ For both parts, say the Taylor series of f about z_0 is

$a_0+a_1(z-z_0)+a_2(z-z_0)^2+\dots$.

a) If $f(z_0)\neq 0$, then $a_0\neq 0$, and $g(z)=\frac{a_0}{z-z_0}+$ (non-princ. part) has a simple pole with residue $a_0=f(z_0)$

b) If $f(z_0)=0$ then $a_0=0$, and the Laurent series of $\frac{f(z)}{z-z_0}$ is a Taylor series $a_1+a_2(z-z_0)+\dots$, so $g(z)$ has a removable singularity, extending to an analytic function with $g(z_0)=$ ~~[illegible]~~ $a_1=f'(z_0)$.

$\boxed{81.3}$ (b) In 73.4 we found the series $\frac{1}{e^z-1}=z^{-1}\left(1-\frac{1}{2}z+\frac{1}{12}z^2+\dots\right)$

So $\frac{1}{z(e^z-1)}=\frac{1}{z^2}\left(1-\frac{1}{2}z+\frac{1}{12}z^2+\dots\right)=\frac{1}{z^2}-\frac{1}{2}\frac{1}{z}+\frac{1}{12}+\dots$

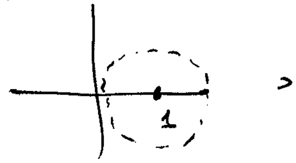
has pole of order $m=2$, with $\mathrm{Res}_{z=0}=-\frac{1}{2}$.

$\boxed{83.4}$ (a) $\cos z$ has zeroes at $z_n=\frac{\pi}{2}+n\pi$, with $\cos'(z_n)=-\sin\left(\frac{\pi}{2}+n\pi\right)=(-1)^{n+1}$, hence $\sec z$ has simple poles with residue $(-1)^{n+1}$ at z_n. Since $z=0$ is not one of the z_n, $z\sec z$ also has a simple pole at z_n, with residue $z_n\cdot(-1)^{n+1}$.

$\boxed{83.5}$ (a) $\tan z$ has poles at the zeroes $z=z_n=\frac{\pi}{2}+n\pi$ of $\cos z$, with residue $\sin(z_n)/\cos'(z_n)=-1$. The contour $|z|=2$ encloses the two poles at $\pm\pi/2$, giving $\int_C \tan z\,dz=(-1+-1)\cdot 2\pi i=-4\pi i$

Add'l Problems

(1) First note that $|z|<1 \Rightarrow z+1$ is in the circle



which is contained in the domain on which the principal value $(z+1)^a = e^{a \,\mathrm{Log}(z+1)}$ is analytic. Its derivatives are $(z+1)^a, \; a(z+1)^{a-1}, \; a(a-1)(z+1)^{a-2}, \dots$

giving Taylor coefficients $\dfrac{f^{(n)}(0)}{n!} = \dfrac{a(a-1)\cdots(a-n+1)}{n!} 1^a = \binom{a}{n}$.

By Taylor's theorem,

$$(z+1)^a = \sum_{n=0}^{\infty} \binom{a}{n} z^n .$$

Note that when a is an integer ≥ 0, $\binom{a}{n} = 0$ for $n > a$, so this infinite series becomes finite and we get the usual binomial theorem.

(2) (a) The Laurent series of $e^{z(w-w^{-1})/2} = e^{zw/2} e^{-zw^{-1}/2}$ (about $w=0$, with z fixed) is the product

$$\left(1 + \left(\tfrac{z}{2}\right) w + \tfrac{1}{2!}\left(\tfrac{z}{2}\right)^2 w^2 + \dots\right)\left(1 - \tfrac{z}{2} w^{-1} + \left(\tfrac{z}{2}\right)^2 w^{-2} + \dots\right).$$

~~There is a contribution~~

Each ~~factor~~ term $\frac{1}{n!}\left(\frac{z}{2}\right)^n w^n$ ~~multiplied~~ in the first factor, multiplied by $\frac{1}{n!}\left(-\frac{z}{2}\right)^n w^{-n}$ in the second, contributes $\dfrac{(-1)^n (z/2)^{2n}}{(n!)^2}$ to the w^0 term. Hence $J_0(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}$.

This converges for all z, giving an entire function $J_0(z)$.

(b) In 68.9 we showed that $J_0(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z \sin\phi)\, d\phi$. Expanding $\cos(z\sin\phi)$ in a Maclaurin series,

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n!)} (\sin\phi)^{2n}\, d\phi = \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n)!} z^{2n} \frac{1}{\pi}\int_0^{\pi} (\sin\phi)^{2n}\, d\phi\right).$$

Equating coefficients with (a), we get

$$\frac{1}{\pi}\int_0^{\pi} (\sin\phi)^{2n}\, d\phi = \frac{(2n)!}{4^n (n!)^2} = \frac{1}{4^n}\binom{2n}{n}.$$

(3) a) In 72.6 we found $\displaystyle \operatorname{Log}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(z-1)^n}{n}$ for $|z-1|<1$

$$= -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n}.$$

Substituting $1/z$ for z and using $\operatorname{Log}(1/z) = -\operatorname{Log}(z)$, this gives

$$\operatorname{Log}(z) = \sum_{n=1}^{\infty} \frac{(1-1/z)^n}{n}$$

for $\left|\frac{1}{z}-1\right| < 1$. Now we need to show that $\operatorname{Re}(z) > \frac{1}{2} \Rightarrow \left|\frac{1}{z}-1\right|<1$.

We have $\displaystyle \left|\frac{1}{z}-1\right|^2 = \left(\frac{1}{z}-1\right)\left(\frac{1}{\bar z}-1\right) = 1+\frac{1}{z\bar z} - \left(\frac{1}{z}+\frac{1}{\bar z}\right)$

$$= \frac{1+z\bar z-(z+\bar z)}{z\bar z} = \frac{|z|^2+1-2\operatorname{Re} z}{|z|^2}.$$

If $\operatorname{Re} z > \frac{1}{2}$, then $1-2\operatorname{Re} z<0$, so $\displaystyle \left|\frac{1}{z}-1\right|^2 < \frac{|z|^2}{|z|^2} = 1$.

b) For positive real x, $\operatorname{Log}(x) = \ln(x)$, so

$$\ln(3) = \sum_{n=1}^{\infty} \frac{(2/3)^n}{n}.$$