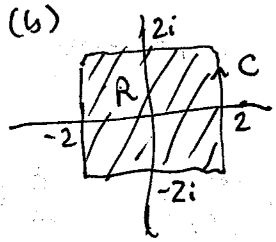
PS 5 Solutions

57.1 (b)



The function $\dfrac{\cos(z)}{z^2+8}$ is analytic on and inside C, since $(-8)^{1/2} = \pm 2i\sqrt{2}$ lie outside the square. By Cauchy integral formula,

$$\int_C \frac{\cos(z)}{z(z^2+8)}\,dz = \int \frac{\cos(z)/(z^2+8)}{z-0}\,dz = 2\pi i\,\frac{\cos(0)}{0^2+8} = \frac{\pi i}{4}.$$

57.5 If z_0 is outside C, both integrals are zero by Cauchy-Goursat. If z_0 is inside C, then

$$\int_C \frac{f'(z)}{z-z_0}\,dz = 2\pi i\, f'(z_0) = \int_C \frac{f(z)}{(z-z_0)^2}\,dz,$$

by Cauchy integral formula (in its extended form, for the second integral).

57.7 Since e^{az} is entire, Cauchy integral formula gives

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i\, e^{a\cdot 0} = 2\pi i,$$

not only for every real a but also for a complex. Putting $z = e^{i\theta}$, we get

$$\int_0^{2\pi} \frac{e^{a(\cos\theta + i\sin\theta)}}{e^{i\theta}}\, i e^{i\theta}\,d\theta = 2\pi i$$

$$\int_0^{2\pi} e^{a\cos\theta}\left(\cos(a\sin\theta) + i\sin(a\sin\theta)\right)d\theta = 2\pi.$$

Taking the real part gives

$$\int_0^{2\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = 2\pi.$$

The integrand is periodic with period 2π, so we can change the limits to $\int_{-\pi}^{\pi}$ (or we could have parametrized C with $-\pi \le \theta \le \pi$ in the first place). The integrand is an even function of θ, so the integral $\int_{-\pi}^{\pi}$ is twice the integral $\int_0^{\pi}$, giving

$$\int_0^{\pi} e^{a\cos\theta} \cos(a\sin\theta)\, d\theta = \pi.$$

57.8 As in §55 (9), $P_n(z)$ is given by

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}\, ds$$

for any C surrounding z. For $z=-1$,

$$\frac{(s^2-1)^n}{(s+1)^{n+1}} = \frac{(s+1)^n(s-1)^n}{(s+1)^{n+1}} = \frac{(s-1)^n}{s+1},$$

so $$P_n(-1) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s-1)^n}{s-(-1)}\, ds = \frac{1}{2^n}\cdot(-1-1)^n = \frac{(-2)^n}{2^n} = (-1)^n,$$

by Cauchy integral formula.

57.10 On $C_R(z_0) = \{z \mid |z-z_0|=R\}$, we have $|z| \le |z_0| + |z-z_0| = |z_0|+R$, so $|f(z)| \le A(|z_0|+R)$. By Cauchy's inequality,

$$|f''(z_0)| \le \frac{2M_R}{R^2} \le 2A\,\frac{|z_0|+R}{R^2} = 2A\left(\frac{|z_0|}{R^2}+\frac{1}{R}\right) \to 0 \quad \text{as } R\to\infty.$$

Since we can choose R as large as we wish, this implies $f''(z_0)=0$ for every z_0, thus $f'(z)$ is constant, and $f(z) = a_1 z + a_2$. But $|f(z)| \le A|z|$ also implies $f(0)=0$, so $a_2 = 0$ and $f(z) = a_1 z$. (We can also conclude that $|a_1| \le A$.)

59.2 Since we assume that $f(z) \neq 0$ for all z in R, $g(z) = \frac{1}{f(z)}$ is analytic on (the interior of) R and not constant. Hence the maximum of $|g(z)|$ occurs on the boundary and not in the interior of R. But $|g(z)| = \frac{1}{|f(z)|}$, so the maximum of $|g(z)|$ occurs where $|f(z)|$ has a minimum.

59.3 If we take $R = \{|z|\le 1\}$ and $f(z)=z$, then the minimum of $|f(z)|$ is 0, and it occurs at $z=0$, in the interior of R and not on boundary.

59.6 I should have assigned 59.5 here! Anyway, $|e^{f(z)}| = e^{u(x,y)}$, and $e^{f(z)}$ is analytic inside R, so $e^{u(x,y)}$ is maximized on (and only on) the boundary. Since e^x is strictly increasing, it follows that $u(x,y)$ is maximized on (and only on) the boundary. The same for $-f(x)$ gives $u(x,y)$ min only on boundary.

Applying this to $u(x,y) = Re(e^z) = e^x\cos(y)$, we need to consider its values on:

i) $y=0,\ x\in[0,1]$ $\quad u(x,0) = e^x$, min $e^0=1$, max $e^1=e$

ii) $x=1,\ y\in[0,\pi]$ $\quad u(1,y) = e\cos(y)$ max $e\cos(0)=e$, min $e\cos(\pi) = -e$.

iii) $y=\pi,\ x\in[0,1]$ $\quad u(x,\pi) = -e^x$, min e^{-1}, max $e^0=1$

iv) $x=0,\ y\in[0,\pi]$ $\quad u(0,y) = \cos(y)$ max $\cos(0)=1$, min $\cos(\pi)=-1$.

The overall maximum is $u(1,0)=e$ at $z=1$, the minimum is $u(1,\pi)=-e$ at $z=1+i\pi$.

61.4 If $z=re^{i\theta}$ with $0<r<1$, then $|z|<1$, so $\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$.

Now $z^n = r^n e^{in\theta} = r^n\cos(n\theta) + i\,r^n\sin(n\theta)$, and

$$\frac{z}{1-z} = \frac{re^{i\theta}}{1-re^{i\theta}} = \frac{(1-re^{-i\theta})re^{i\theta}}{(1-re^{i\theta})(1-re^{-i\theta})} = \frac{r\cos\theta - r^2 + ir\sin\theta}{1+r^2-r(e^{i\theta}+e^{-i\theta})}$$

$$= \frac{r\cos\theta - r^2 + ir\sin\theta}{1+r^2-2r\cos\theta}$$

Taking $Re(-)$ and $Im(-)$ in $\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$ gives

$$\sum_{n=1}^{\infty} r^n\cos(n\theta) = \frac{r\cos\theta - r^2}{1+r^2-2r\cos\theta}, \qquad \sum_{n=1}^{\infty} r^n \sin(n\theta) = \frac{r\sin\theta}{1+r^2-2r\cos\theta}$$

65.1 Set $z=z^2$ in Example 2 to get $\cosh(z^2) = \sum_{n=0}^{\infty}\frac{z^{4n}}{(2n)!}$, then multiply by z: $\quad z\cosh(z^2) = \cancel{z}\sum_{n=0}^{\infty}\frac{z^{4n+1}}{(2n)!}$ (abs. convergent for all z).

65.2 (a) $\left(\frac{d}{dz}\right)^n e^z \Big|_{z=1} = e^z\Big|_{z=1} = e$, so

$$e^z = \sum_{n=0}^{\infty} \frac{e}{n!}(z-1)^n = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

(b) Substitute $z \mapsto z-1$ in $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ to get

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!}, \qquad e^z = e^1 e^{z-1} = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

65.5 Starting with $\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$, we get

$\sinh(z+\pi i) = -\sinh(z) = -\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$, and then

substituting $z-\pi i$ for z gives

$$\sinh(z) = -\sum_{n=0}^{\infty} \frac{(z-\pi i)^{2n+1}}{(2n+1)!}$$

(with ∞ radius of convergence, because $\sinh(z)$ is entire).

65.11 $\frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}} = \frac{1}{4z} + \frac{1}{16}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n = \frac{1}{4z} + \frac{1/16}{1-z/4}$ $(0<|z|<4)$

$$= \frac{1 - z/4 + 4z/16}{4z(1-z/4)} = \frac{1}{4z-z^2}.$$

Add'l Problems

1. $\int_C \frac{1}{z(z^2-1)}\,dz = \int_C \frac{-1}{z} + \frac{-1/2}{z-1} + \frac{1/2}{z+1}\,dz$

$$= \int_C \frac{-1}{z}\,dz + \int_C \frac{-1/2}{z-1}\,dz + \int \frac{1/2}{z+1}\,dz.$$

The first integral is $\frac{-1}{2\pi i}$ if C surrounds 0, otherwise zero.
The second is $\frac{-1}{4\pi i}$ if C surrounds 1, otherwise zero.
The third is $\frac{1}{4\pi i}$ if C surrounds -1, otherwise zero.
The possible values are all sums of subsets of $\frac{1}{2\pi i}\{-1, \pm\frac{1}{2}\}$,
that is $\frac{1}{2\pi i}\{-\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}\}$, depending on which subset
of $\{0, \pm 1\}$ the contour C encloses.

2. This is actually a special case of 57.7. I didn't notice that when assigning the problems.