PS5 Add'l Problem 1 Solution (corrected)

The partial fraction expansion is

$$\frac{1}{z(z^2-1)} = \frac{1}{z(z-1)(z+1)} = \frac{-1}{z} + \frac{1/2}{z-1} - \frac{1/2}{z+1} .$$

Therefore

$$\int_C \frac{1}{z(z^2-1)}\,dz = -1\int_C \frac{1}{z}\,dz + \frac{1}{2}\int_C \frac{1}{z-1}\,dz - \frac{1}{2}\int_C \frac{1}{z+1}\,dz .$$

The first integral is

$$-1\int_C \frac{1}{z}\,dz = \begin{cases} -2\pi i & \text{if } C \text{ encloses } 0 \\ 0 & \text{if not,} \end{cases}$$

by Cauchy integral formula and Cauchy-Goursat theorem.

Similarly, the other terms are

$$\frac{1}{2}\int_C \frac{1}{z-1}\,dz = \begin{cases} \pi i & \text{if } C \text{ encloses } 1 \\ 0 & \text{if not} \end{cases}$$

$$-\frac{1}{2}\int_C \frac{1}{z-1}\,dz = \begin{cases} -\pi i & \text{if } C \text{ encloses } -1 \\ 0 & \text{if not} \end{cases}$$

The 8 possible combinations for which of the points $\{0,-1,1\}$ C encloses lead to possible values of the integral of the form $2\pi i \cdot$ (sum of a subset of $\{-1, -\frac{1}{2}, \frac{1}{2}\}$). Since $-\frac{1}{2}+\frac{1}{2}=0$, not all subsets result in distinct values. All the possibilities are $2\pi i \cdot \{-\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}\}$, with -1, $-\frac{1}{2}$ and 0 each occuring for two of the 8 configurations of the contour.