PS 4 Solutions

30.4 Method 1: since e^z and z^2 are entire, so is e^{z^2}, with $\frac{d}{dz}e^{z^2} = 2ze^{z^2}$ by the chain rule.

Method 2: $e^{z^2} = u + iv$ with $u(x,y) = e^{x^2-y^2}\cos(2xy)$, $v = e^{x^2-y^2}\sin(2xy)$. Now verify Cauchy-Riemann equations.

30.7 Since $|e^z| = e^{Re(z)}$, $|e^{-2z}| = e^{-2x} < 1$ if and only if $2x > 0$, if and only if $Re(z) = x > 0$.

30.10 (a) $e^{x+iy} = e^x(\cos y + i\sin y)$ is real $\Leftrightarrow$ $\sin y = 0$, i.e. $y = Im(z) = n\pi$ for integer n

(b) e^{x+iy} is imaginary $\Leftrightarrow$ $\cos y = 0$ $\Leftrightarrow$ $Im(z) = (n+\frac{1}{2})\pi$ for integer n

33.6 $e^{\log z} = z \underset{\frac{d}{dz}}{\Rightarrow} e^{\log z}\log(z)' = 1 \Rightarrow \log(z)' = \frac{1}{e^{\log z}} = \frac{1}{z}$.

This calculation is valid for any branch of $\log(z)$ on the domain where that branch is analytic.

33.9 The values of $\log(e^{x+iy}) = \log(e^x e^{iy})$ are $\ln(r) + i\theta$ for θ such that $e^{i\theta} = e^{iy}$, i.e. $\theta = y + 2\pi n$ for integer n. If $\alpha < y < \alpha + 2\pi$, then $\theta = y$ on the branch $\log(re^{i\theta}) = \ln(r) + i\theta$ for $\alpha < \theta < \alpha + 2\pi$.

33.11 $\ln(x^2+y^2)$ is the real part of any branch of $\log(z)$. Since $\log(z)$ has a branch analytic at any $z \neq 0$, it follows that $\ln(x^2+y^2)$ is harmonic on $(x,y) \neq (0,0)$. This can also be checked directly by evaluating u_{xx} and u_{yy}, where $u(x,y) = \ln(x^2+y^2)$.

34.1 Since the values of $\log(z)$ are $Log(z) + 2n\pi i$ for n integer, and $\log(z_1z_2) = \log(z_1) + \log(z_2)$ as multiple valued functions, we see that $Log(z_1z_2)$ and $Log(z_1) + Log(z_2)$ must differ by an integer multiple $2N\pi i$ of $2\pi i$. Now $Im(Log(z_1z_2))$ is in $(-\pi, \pi]$ by the definition of $Log(z)$; hence $Im(Log(z_1) + Log(z_2))$ is in $(-2\pi, 2\pi]$, so $Im(Log(z_1z_2))$ and $Im(Log(z_1) + Log(z_2))$ differ by less than 3π, which implies $N \in \{-1, 0, 1\}$.

34.5 The values of $z^{1/n}$ are $\sqrt[n]{r}\, e^{i\psi}$, where $\psi = \frac{\theta + 2k\pi}{n}$ for integers $k = 0, 1, \ldots, n-1$. Then the values of $\log(z^{1/n})$ are

$$\ln(\sqrt[n]{r}) + i(\psi + 2\pi p) \qquad (p \text{ integer})$$

$$= \frac{1}{n}\ln(r) + i\,\frac{\theta + 2(pn+k)\pi}{n}. \qquad (p \text{ integer},\ k = 0, \ldots, n-1)$$

Clearly $pn + k = q$ is an integer, and it can be any integer, since we can take p, k to be the quotient and remainder of $q \div n$. So we can express the values of $\log(z^{1/n})$ as

$$\frac{1}{n}\ln(r) + i\,\frac{\theta + 2q\pi}{n} \qquad (q \text{ integer}),$$

which are the same as the values of $\frac{1}{n}\log(z)$.

36.8 In fact all that matters is the value we choose for $\log(z)$, and we can take it from any branch, provided we use the same branch every time we evaluate $\log(z)$. Then:

a) $z^{c_1} z^{c_2} = e^{c_1 \log(z)} e^{c_2 \log(z)} = e^{(c_1 + c_2)\log(z)} = z^{c_1 + c_2}$

b) $z^{c_1}/z^{c_2} = e^{c_1 \log(z)} e^{-c_2 \log(z)} = e^{(c_1 - c_2)\log(z)} = z^{c_1 - c_2}$

c) $(z^c)^n = (e^{c \log(z)})^n = e^{cn \log(z)} = z^{cn}$

38.11
If $z = x + iy$, then $\cos(\bar{z}) = \cos(x - iy) = \cos(x)\cosh(-y) - i\sin(x)\sinh(-y)$
$= \cos(x)\cosh(y) + i\sin(x)\sinh(y)$.

With $u = \cos(x)\cosh(y)$ and $v = \sin(x)\sinh(y)$, we get

$$u_x = -\sin(x)\cosh(y) \qquad v_x = \cos(x)\sinh(y)$$

$$u_y = \cos(x)\sinh(y) \qquad v_y = \sin(x)\cosh(y).$$

Then Cauchy-Riemann eqn's $u_x = v_y$, $u_y = -v_x$ imply
$\sin(x)\cosh(y) = 0$, $\cos(x)\sinh(y) = 0$.

Now $\cosh(y)$ is never 0, so these equations imply $\sin(x) = 0$, hence $x = n\pi$ (n integer) and $\cos(x) = \pm 1$. This forces $\sinh(y) = 0$, i.e. $e^y = e^{-y}$, $e^{2y} = 1$, $y = 0$. Thus the C-R equations hold only at $x = n\pi$, $y = 0$, i.e. $z = n\pi$. In particular, they do not hold

on a neighborhood of any z, so $\cos(\bar{z})$ is nowhere analytic.
The calculations and reasoning for $\sin(\bar{z})$ are similar, using

$$\sin(\bar{z}) = \sin(x)\cosh(y) - i\cos(x)\sinh(y).$$

40.2 a) $\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$, so $\sin(z) = 2$ iff $\cos(x)\sinh(y) = 0$ and $\sin(x)\cosh(y) = 2$. If $\cos(x) \neq 0$, then $\sinh(y) = 0$, which implies $y = 0$, $\cosh(y) = 1$, $\sin(x) = 2$, which is impossible. So we must have $\cos(x) = 0$, hence $\sin(x) = \pm 1$. But $\cosh(y) > 0$, so we must have $\sin(x) = 1$, $x = (2n + \frac{1}{2})\pi$, and $\cosh(y) = 2$, that is $e^y + e^{-y} = 4$, $e^{2y} - 4e^y + 1 = 0$,
$e^y = \dfrac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$, $y = \ln(2 \pm \sqrt{3})$, thus

$$z = (2n + \tfrac{1}{2})\pi + i\ln(2 \pm \sqrt{3}).$$

Note that $(2+\sqrt{3})(2-\sqrt{3}) = 1$, so $\ln(2-\sqrt{3}) = \ln\left(\frac{1}{2+\sqrt{3}}\right) = -\ln(2+\sqrt{3})$, so we can also express this as

$$z = (2n + \tfrac{1}{2})\pi \pm i\ln(2+\sqrt{3}).$$

b)
$$\begin{aligned}\sin^{-1}(2) &= -i\log(2i + (-3)^{1/2}) \\ &= -i\log((2 \pm \sqrt{3})i) \\ &= -i\left(\ln(2 \pm \sqrt{3}) + i(\tfrac{\pi}{2} + 2\pi n)\right)\end{aligned}$$

Since $\operatorname{Arg}(\cancel{\quad}(2+\sqrt{3})i) = \operatorname{Arg}((2-\sqrt{3})i) = \pi/2$ as $\cancel{\quad}\ 2 \pm \sqrt{3}$ are both real and positive. The above is equal to

$$(2n + \tfrac{1}{2})\pi \pm i\ln(2+\sqrt{3})$$

using $\ln(2-\sqrt{3}) = -\ln(2+\sqrt{3})$ as before.

42.4 $\displaystyle\int_0^\pi e^{(1+i)x}\,dx = \left.\frac{e^{(1+i)x}}{1+i}\right]_0^\pi = \frac{e^{(1+i)\pi} - 1}{1+i} = \frac{-e^\pi - 1}{1+i} = (1-i)\frac{-e^\pi - 1}{2}.$

Hence $\displaystyle\int_0^\pi e^x\cos x\,dx = \operatorname{Re}\int_0^\pi e^{(1+i)x}\,dx = -(e^\pi + 1)/2$

$\displaystyle\int_0^\pi e^x\sin x\,dx = \operatorname{Im}\int_0^\pi e^{(1+i)x}\,dx = (e^\pi + 1)/2$

43.5 [In this exercise we prove that the chain rule is valid for an analytic function $f(z)$ composed with a complex valued function $z(t)$ of a real variable t.]

Let $f(x+iy) = u(x,y) + i\,v(x,y)$ and $z(t) = x(t) + i\,y(t)$.

Then $w(t) = f(z(t)) = u(x(t),y(t)) + i\,v(x(t),y(t))$, so

$$\begin{aligned}
w'(t) &= u_x(x(t),y(t))\,x'(t) + u_y(x(t),y(t))\,y'(t) \\
&\qquad + i\big(v_x(x(t),y(t))\,x'(t) + v_y(x(t),y(t))\,y'(t)\big) \\
&= u_x(x(t),y(t))\,(x'(t)+iy'(t)) \\
&\qquad + u_y(x(t),y(t))\,(y'(t)-ix'(t)) \qquad \text{by Cauchy-Riemann} \\
&= \big(u_x(x(t),y(t)) + iu_y(x(t),y(t))\big)(x'(t)+iy'(t)) \\
&= f'(z(t))\,z'(t).
\end{aligned}$$

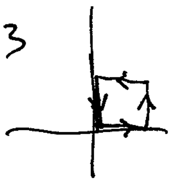
This is valid at any t where $x'(t)$, $y'(t)$ and the ~~relevant~~ first partial derivatives of u, v at $(x(t), y(t))$ exist, and Cauchy-Riemann holds. In particular it is valid at t_0 if $z'(t_0)$ exists and $f'(z(t_0))$ exists (it is not actually necessary that f be analytic at $z(t_0)$).

46.1 a) $$\int_0^\pi \frac{2e^{i\theta}+2}{2e^{i\theta}}(2ie^{i\theta})\,d\theta = \int_0^\pi 2i(e^{i\theta}+1)\,d\theta = 2i\left(\left[\frac{e^{i\theta}}{i}\right]_0^\pi + \pi\right)$$
$$= 2i\left(\frac{-2}{i}+\pi\right) = -4 + 2\pi i$$

b) Same integral as in (a) but with limits $\int_\pi^{2\pi}$, giving
$$2i\left(\left[\frac{e^{i\theta}}{i}\right]_\pi^{2\pi} + \pi\right) = 4 + 2\pi i$$

c) Add (a) and (b), or integrate with limits $\int_0^{2\pi}$, to get $4\pi i$

46.3



$$\int_0^1 \pi e^{\pi x}\,dx + \int_0^1 \pi e^{\pi(1+ix)}\,i\,dx + \int_0^1 \pi e^{\pi((1-x)+i)}(-dx) + \int_0^1 \pi e^{-\pi i(1-x)}(-i\,dx)$$
$$= e^\pi - 1 \quad + \quad 2e^\pi \quad + \quad e^\pi - 1 \quad + \quad -2$$
$$= 4e^\pi - 4$$

46.8 Parametrize the circle as $z = Re^{i\theta}$ with $-\pi \le \theta \le \pi$, so that $\mathrm{Arg}(z) = \theta$, and $\mathrm{Log}(z) = \ln(R) + i\theta$. This gives

$$\int_C z^{a-1}\,dz = \int_{-\pi}^{\pi} e^{(a-1)(\ln(R)+i\theta)} \cdot iRe^{i\theta}\,d\theta$$

$$= iRe^{(a-1)\ln(R)} \int_{-\pi}^{\pi} e^{ia\theta}\,d\theta$$

$$= iR^a \left.\frac{e^{ia\theta}}{ia}\right]_{-\pi}^{\pi} = iR^a \frac{e^{ia\pi} - e^{-ia\pi}}{ia} = i\frac{2R^a}{a}\sin(\pi a)$$

46.13 If $n \neq 0$, then $(z-z_0)^{n-1}$ is the derivative of the ~~entire~~ function $\frac{(z-z_0)^n}{n}$ on ~~[illegible]~~ $z \neq z_0$ (in case $n<0$), and then $\int_{C_0}(z-z_0)^{n-1}\,dz = 0$ by the fundamental theorem of calculus (Theorem in § 48).

For $n=0$, we'll actually calculate it:

$$\int_{C_0} (z-z_0)^{-1}\,dz = \int_{-\pi}^{\pi} R^{-1}e^{-i\theta}\cdot(Rie^{i\theta})\,d\theta = \int_{-\pi}^{\pi} i\,d\theta = 2\pi i$$

47.5 Since $\mathrm{Log}(z) = \ln(R) + i\theta$ with $-\pi < \theta \le \pi$, we have

$$|\mathrm{Log}(z)| \le |\ln(R)| + |i\theta| \le \ln(R) + \pi,$$ hence

$$\left|\frac{\mathrm{Log}(z)}{z^2}\right| = \frac{|\mathrm{Log}(z)|}{R^2} \le \frac{\ln(R)+\pi}{R^2}.$$

The length of the contour C_R is $2\pi R$, so

$$\left|\int_{C_R} \frac{\mathrm{Log}(z)}{z^2}\,dz\right| \le 2\pi R\left(\frac{\ln(R)+\pi}{R^2}\right) = 2\pi\left(\frac{\ln(R)+\pi}{R}\right).$$

By L'Hospital, $\lim_{R\to\infty} \frac{\ln(R)}{R} = \lim_{R\to\infty}\frac{1/R}{1} = 0$, hence $\lim_{R\to\infty} 2\pi\left(\frac{\ln(R)+\pi}{R}\right) = 0$.

49.3 See 46.13

53.4 a) The lower leg is $\int_{-a}^{a} e^{-x^2}\,dx = 2\int_0^a e^{-x^2}\,dx$, since e^{-x^2} is an even function. The upper leg is

$$\int_{-a}^{a} e^{-(bi-x)^2}(-dx) = -\int_{-a}^{a} e^{b^2-x^2}\, e^{2bix}\,dx .$$

Now $e^{2bix} = \cos 2bx + i\sin 2bx$. The imaginary part is an odd function, and cancels, while the real part is even, giving

$$-2e^{b^2}\int_{-a}^{a} e^{-x^2}\cos 2bx\,dx .$$

So the upper and lower legs contribute

$$2\int_0^a e^{-x^2}\,dx - 2e^{b^2}\int_{-a}^{a} e^{-x^2}\cos 2bx\,dx .$$

b) The right leg is

$$\int_0^b e^{-(a+iy)^2}\, i\,dy = \int_0^b e^{y^2-a^2-2iay}\, i\,dy = ie^{-a^2}\int_0 e^{y^2}e^{-i2ay}\,dy .$$

The left leg is

$$\int_b^0 e^{-(-a+iy)^2}\, i\,dy = -\int_0^b e^{-(-a+iy)^2}\, i\,dy ,$$

which is − the previous integral with a changed to −a, thus $-ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}\,dy$.

~~By Cauchy-Goursat, the sum of all 4 legs is zero, since e^{-z^2} is entire, giving, when we take the real part,~~

The two vertical legs together give

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}\,dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}\,dy = -ie^{-a^2}\int_0 e^{y^2}\left(e^{i2ay}-e^{-i2ay}\right)dy$$

$$= -2e^{-a^2}\int_0^b e^{y^2}\sin(2ay)\,dy$$

The sum of all 4 legs is zero by Cauchy-Goursat:

$$2\int_0^a e^{-x^2}\,dx - 2e^{b^2}\int_0^a e^{-x^2}\cos(2bx)\,dx - 2e^{-a^2}\int_0^b e^{y^2}\sin(2ay)\,dy ,$$

or

$$\int_0^a e^{-x^2}\cos(2bx)\,dx = e^{-b^2}\int_0^a e^{-x^2}\,dx - e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin(2ay)\,dy .$$

Since $|\sin(2ay)| \leq 1$, $\left| \int_0^b e^{y^2} \sin(2ay)\,dy \right| \leq \int_0^b e^{y^2}$, a constant which does not depend on a. Hence the last term $\to 0$ as $a \to \infty$, leaving

$$\int_0^\infty e^{-x^2} \cos(2bx)\,dx = e^{-b^2} \int_0^\infty e^{-x^2} dx = \frac{e^{-b^2}\sqrt{\pi}}{2}.$$

Note that the derivation assumes $b>0$, but both sides are even functions of b, so the formula also holds for $b<0$.

Add'l Problems

1. Because the function we are integrating, $f(z) = \pi e^{\pi \bar{z}}$, is not analytic.

2. If we had $f'(z) = 1/z$ for all $z \neq 0$, then the theorem in §48 would imply that $\int_C \frac{1}{z}\,dz = 0$ on any ~~each~~ closed contour C not passing through 0. But Ex. 46.13 gives a counterexample to this (taking $z_0 = 0$ and any $R>0$).