PS 3 Solutions

18.1 (c). Since $|\bar{z}^2/z| = |\bar{z}|^2/|z| = |z|$, given $\varepsilon > 0$ we can take $\delta = \varepsilon$ to get $|z - \underset{z_0=0}{\underset{\uparrow}{0}}| < \delta \Rightarrow |\bar{z}^2/z - \underset{\text{value of the limit}}{\underset{\uparrow}{0}}| < \varepsilon$.

18.5 Since $|\bar{z}| = |z|$, we have $|f(z)| = 1$ for all z.
If z is real, then $z/\bar{z} = 1$, and if z is imaginary, then $z/\bar{z} = -1$, so $(z/\bar{z})^2 = 1$. Thus $f(z) \to 1$ as $z \to 0$ along either axis. However, along the line $y = x$, $z = (x, x) = x(1+i)$,
$$z/\bar{z} = \frac{x(1+i)}{x(1-i)} = \frac{1+i}{1-i} = \frac{\sqrt{2}\, e^{i\pi/4}}{\sqrt{2}\, e^{-i\pi/4}} = e^{i\pi/2} = i, \quad \text{so } (z/\bar{z})^2 = -1.$$

18.9 Given $\varepsilon > 0$, using $\lim_{z \to z_0} f(z) = 0$ we can find a δ such that $z \neq z_0$, $|z - z_0| < \delta \Rightarrow |f(z)| < \varepsilon/M$ (here we are applying the definition with ε/M in the role of ε). Then
$|z - z_0| < \delta \Rightarrow |f(z)g(z)| = |f(z)| \cdot |g(z)| < \frac{\varepsilon}{M} \cdot M = \varepsilon$,
so $\lim_{z \to z_0} f(z) g(z) = 0$

18.11 (a) If $c = 0$, then $a \neq 0$ and $d \neq 0$, since $ad - bc \neq 0$. In this case $T(z) = \frac{az+b}{d}$ and $\frac{1}{T(1/z)} = \frac{d}{a/z + b} = \frac{zd}{a + bz}$.
This is continuous at $z = 0$ with value 0, so $\lim_{z \to 0} \frac{1}{T(1/z)} = 0$, which means $\lim_{z \to \infty} T(z) = \infty$.

(b) If $c \neq 0$, then $\lim_{z \to \infty} T(z) = \lim_{z \to 0} T\left(\frac{1}{z}\right) = \lim_{z \to 0} \frac{a/z + b}{c/z + d} = \lim_{z \to 0} \frac{a + bz}{c + dz} = a/c$,
and $\lim_{z \to -d/c} \frac{1}{T(z)} = \lim_{z \to -d/c} \frac{cz + d}{az + b}$. Now $a\left(\frac{-d}{c}\right) + b \neq 0$ since $ad - bc \neq 0$,
so $\lim_{z \to -d/c} \frac{cz + d}{az + b} = 0$, which means $\lim_{z \to -d/c} T(z) = \infty$.

20.3 (a) Starting with $f'(z)=0$ for $f(z)=z^0=1$, and $f'(z)=1$ for $f(z)=z$, the product rule implies, by induction, that

$$\frac{d}{dz}(z^n) = \frac{d}{dz}(z\cdot z^{n-1}) = z(n-1)z^{n-2} + z^{n-1} = n z^{n-1}.$$

Then the sum rule gives

$$\frac{d}{dz}(a_0 + a_1 z + \dots + a_n z^n) =$$
$$a_1 + 2a_2 z + \dots + n a_n z^{n-1}.$$

(b) The constant term of the k^{th} derivative of $P(z)$, i.e. $p^{[k]}(0)$, comes from the z^k term of $P(z)$:

$$p^{[k]}(0) = \left(\frac{d}{dz}\right)^k a_k z^k.$$

The higher derivatives of z^k are

$$k z^{k-1},\quad k(k-1)z^{k-2},\quad \dots,\quad k(k-1)(k-2)\cdots(1)z^0 = k!,$$

so $p^{[k]}(0) = k!\, a_k$, $\quad a_k = \dfrac{p^{[k]}(0)}{k!}$

20.4 Defining $\Delta z = z - z_0$, $\Delta f = f(z) - f(z_0)$, $\Delta g = g(z) - g(z_0)$, we have

$$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim_{\Delta z\to 0} \Delta f/\Delta z}{\lim_{\Delta z\to 0} \Delta g/\Delta z}.$$

Since the limits in the numerator and denominator exist, and the one in the denominator is non-zero, this is equal to $\lim_{\Delta z\to 0} \frac{\Delta f}{\Delta g}$. But $f(z_0)=g(z_0)=0$, so $\Delta f = f(z)$ and $\Delta g = g(z)$.

So this just says $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{f'(z_0)}{g'(z_0)}$.

Remark This problem proves a weak form of L'Hopital's rule, in which we assume that $f'(z_0)$ and $g'(z_0)$ exist and $g'(z_0)\neq 0$. The stronger form is that if $f(z_0)=g(z_0)=0$ and $f'(z)$, $g'(z)$ exist in a neighborhood of z_0, then

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{f'(z)}{g'(z)},$$

even if $g'(z_0)=0$, provided the limit on the right hand side exists. This version is also valid in the complex case.

20.8(a) We are to show that $\lim_{z \to z_0} \frac{Re(z) - Re(z_0)}{z - z_0}$ does not exist, for any z_0. Now $Re(z) - Re(z_0) = Re(z - z_0)$. On the line $z = x + iy_0$, where $z - z_0$ is real, therefore, $\frac{Re(z) - Re(z_0)}{z - z_0} = 1$.
On the line $z = x_0 + iy$, where $z - z_0$ is imaginary, $\frac{Re(z) - Re(z_0)}{z - z_0} = 0$. Hence the limit does not exist, since $\frac{Re(z) - Re(z_0)}{z - z_0}$ takes both values 0 and 1 in every disk $D_\delta(z_0)$.

20.9 Let $w = f(z)$, $z_0 = 0$, $w_0 = f(z_0) = 0$, so $\Delta z = z - z_0 = z$,
$\Delta w = f(z) - f(z_0) = f(z) = \bar{z}^2/z$, $\Delta w / \Delta z = \bar{z}^2/z^2 = (\bar{z}/z)^2$.
In Ex. 18.5 we already showed that this function is 1 on the Re and Im axes and -1 on the line $y = x$. Hence $f'(0)$ does not exist — even though, if we put $f(z) = u(x,y) + i\,v(x,y)$, then $u(x,0) = x$, $v(x,0) = 0$, $u(0,y) = 0$, $v(0,y) = y$, so u_x, u_y, v_x, v_y exist at $(0,0)$, and $u_x = v_y = 1$, $u_y = -v_x = 0$, so the Cauchy-Riemann equations hold. (The trouble is that u_x, u_y, v_x, v_y aren't continuous at 0).

24.2(d) With $u = \cos x \cosh y$ and $v = -\sin x \sinh y$, we have

$$u_x = -\sin x \cosh y = v_y$$
$$u_y = \cos x \sinh y = -v_x .$$

Since u_x, u_y, v_x, v_y are continuous and satisfy Cauchy-Riemann, $f'(z)$ exists, and is given by $f'(z) = -\sin x \cosh y - i \cos x \sinh y$.
A similar calculation with $u = -\sin x \cosh y$, $v = -\cos x \sinh y$ gives $f''(z) = -\cos x \cosh y + i \sin x \sinh y = -f(z)$.

24.4 (b) Check polar Cauchy-Riemann:

$$r u_r = r\left(-e^{-\theta} \sin(\ln r) \tfrac{1}{r}\right) = -e^{-\theta} \sin(\ln r) = v_\theta$$
$$r v_r = r\left(e^{-\theta} \cos(\ln r) \tfrac{1}{r}\right) = e^{-\theta} \cos(\ln r) = -u_\theta .$$

Hence $f'(z)$ exists and is given by $(u_r + i v_r) e^{-i\theta} = \frac{-e^{-\theta}\sin(\ln r) + i e^{-\theta}\cos(\ln r)}{r e^{i\theta}}$
$= i f(z)/z$.

24.6 By Cauchy-Riemann, $v_x = -u_y$, so the formulas in Ex. 24.5 give

$$f'(z_0) = u_x - iu_y = u_r \cos\theta - u_\theta \frac{\sin\theta}{r} - i\left(u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right)$$

$$= \left(u_r - i\frac{u_\theta}{r}\right)(\cos\theta - i\sin\theta).$$

By polar Cauchy-Riemann, $u_\theta/r = -v_r$, so this becomes

$$(u_r + iv_r)e^{-i\theta}$$

[We got the same thing in class by a simpler method.]

24.7 (a) Use $u_r = \frac{v_\theta}{r}$, $v_r = \frac{-u_\theta}{r}$ to get

$$(u_r + iv_r)e^{-i\theta} = (v_\theta - iu_\theta)\frac{1}{re^{i\theta}} = \frac{-i}{z_0}(u_\theta + iv_\theta).$$

(b) If $f(z) = \frac{1}{z} = \frac{1}{r}e^{-i\theta}$, then $u = \frac{1}{r}\cos\theta$, $v = -\frac{1}{r}\sin\theta$,

$u_\theta = -\frac{1}{r}\sin\theta$, $v_\theta = -\frac{1}{r}\cos\theta$,

$$-\frac{i}{z}(u_\theta + iv_\theta) = -\frac{i}{z}\left(-\frac{1}{r}(\sin\theta + i\cos\theta)\right) = \frac{-i}{z}\left(\frac{-ie^{-i\theta}}{r}\right)$$

$$= \frac{-i}{z}\cdot\frac{-i}{z} = \frac{i^2}{z^2} = -1/z^2$$

26.1 (c) $u = e^{-y}\sin x$, $v = -e^{-y}\cos x$

$u_x = e^{-y}\cos x = v_y$ ✓

$u_y = -e^{-y}\sin x = -v_x$ ✓

26.2 (b) $u_x = 2y$ $\quad v_y = -2y$

$u_y = 2x$ $\quad v_x = 2x$

So Cauchy-Riemann eq'ns $2y = -2y$, $2x = -2x$ hold only at $z = 0$. Our function is differentiable at 0, with $f'(0) = 0$, but not analytic, since it's not differentiable on a <u>neighborhood</u> of 0.

26.7 $f(z)$ real means $v = 0$. Then Cauchy-Riemann gives $u_x = v_y = 0$, $u_y = -v_x = 0$, so u is constant (as D is connected) and therefore $f = u + i\cdot 0$ is constant.

27.2 The gradient $\vec{a} = (u_x(z_0), u_y(z_0))$ is the normal vector to the level curve $u(x,y) = c_1$, and the gradient $\vec{b} = (v_x(z_0), v_y(z_0))$ is normal to the level curve $v(x,y) = c_2$. By Cauchy-Riemann, $\vec{a} = (b_2, -b_1)$, where $\vec{b} = (b_1, b_2)$, and furthermore $f'(z_0) = a_1 + i a_2$, so $\vec{a} \neq 0$ and therefore $\vec{b} \neq 0$. Since the gradients are non-zero and perpendicular, the two level curves are smooth (i.e. they have tangent lines at z_0) and their tangent lines are perpendicular.

27.3 If $f(z) = z^2$, then $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$.
The level curves $x^2 - y^2 = c_1$, or $(x+y)(x-y) = c_1$ are hyperbolas asymptotic to the lines $x = \pm y$, if $c_1 \neq 0$.
The level curves $2xy = c_2$ are hyperbolas asymptotic to the x and y axes, if $c_2 \neq 0$.
If $c_1 = 0$, then $x^2 - y^2 = 0$ is the union of the lines $x = \pm y$, which are $\perp$ to the hyperbolas $2xy = c_2$; if $c_2 = 0$, then $2xy = 0$ is the union of the x and y axes, which are $\perp$ to the hyperbolas $x^2 - y^2 = c_1$.
However, neither $2xy = 0$ nor $x^2 - y^2 = 0$ has a tangent line at the origin; this is consistent with Ex. 27.2 because $f'(z) = 2z = 0$ at the origin.

Add'l Problem 1.
$z^2 + 2z + 2 = (z+1-i)(z+1+i)$ and $(1+i)z + 2 = (z+1-i)(1+i)$,
so $$\lim_{z \to -1+i} \frac{(1+i)z+2}{z^2+2z+2} = \lim_{z \to -1+i} \frac{1+i}{z+1+i} = \frac{1+i}{2i} = \frac{1}{2} - i\frac{1}{2}$$

2. Using Theorem 18.3, $f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$. Both $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$ are continuous for $(x,y) \neq (0,0)$, so $f(z)$ is continuous for $z \neq 0$.

3. On the domain $D = (r > 0, \ -\pi < \theta < \pi)$,

$g(z) = \ln r + i\theta$ has $u_r = \frac{1}{r}$, $r u_r = 1 = v_\theta$

$u_\theta = 0$, $v_r = 0$, so $-r v_r = 0 = u_\theta$.

The partial derivatives are continuous on D and satisfy Cauchy-Riemann, so $g'(z) = (u_r + i v_r) e^{-i\theta}$

$= \frac{1}{r} e^{-i\theta} = \frac{1}{re^{i\theta}} = \frac{1}{z}$, by Ex. 24.6.

Now if $x = \mathrm{Re}(z) > 0$, then $-\pi/2 < \mathrm{Arg}(z) < \pi/2$, so $-\pi < \mathrm{Arg}(z^2) < \pi$, i.e. z^2 is not on the $-\mathrm{Re}$ axis, and so z^2+1 is also not on the $-\mathrm{Re}$ axis, that is, $z^2+1 \in D$.

Hence $G(z) = g(z^2+1)$ is analytic for $x > 0$ with derivative given by the chain rule:

$$G'(z) = g'(z^2+1) \cdot 2z = \frac{2z}{z^2+1}$$