PS 2 Solutions

5.4 Let $z = x + iy$. We have

$$(|x| - |y|)^2 = |x|^2 + |y|^2 - 2|xy| \geq 0,$$

and adding $|x|^2 + |y|^2 + 2|xy|$ to both sides,

$$2(|x|^2 + |y|^2) \geq |x|^2 + |y|^2 + 2|xy| = (|x| + |y|)^2.$$

Now take $\sqrt{\ }$ to get

$$\sqrt{2}\,|z| = \sqrt{2}\sqrt{x^2+y^2} \geq |x| + |y| = |\mathrm{Re}\, z| + |\mathrm{Im}\, z|.$$

[Note that you would most likely discover this reasoning by working backwards from $\sqrt{2}\,|z| \overset{?}{\geq} |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$, but you must present it in a logical sequence that ends with the desired inequality.]

6.10 (a) Let $z = x + iy$. Then $\bar{z} = x - iy$, so $z = \bar{z}$ iff $y = -y$ iff $y = 0$ iff z is real.

(b) Since $\bar{z}^2 = \overline{z^2}$, part (a) gives $\bar{z}^2 = z^2$ iff z^2 is real. This is equivalent to z being either real (if $z^2 \geq 0$) or pure imaginary (if $z^2 \leq 0$).

6.11 The basis step is $n = 1$, for which both (a) and (b) are trivial. Assume (a) and (b) hold for $n-1$ by induction. For (a) we have

$$\overline{z_1 + z_2 + \ldots + z_n} = \overline{(z_1 + z_2 + \ldots + z_{n-1})} + \bar{z}_n \quad \text{by §6 (2)}$$
$$= \bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_{n-1} + \bar{z}_n \quad \text{by induction.}$$

For (b),

$$\overline{z_1 z_2 \cdots z_n} = \overline{z_1 z_2 \cdots z_{n-1}} \cdot \bar{z}_n \quad \text{by §6 (4)}$$
$$= \bar{z}_1\, \bar{z}_2 \cdots \bar{z}_n \quad \text{by induction.}$$

6.14 Since $x = \mathrm{Re}\, z = (z + \bar{z})/2$ and $y = \mathrm{Im}\, z = (z - \bar{z})/2i$, we can write $x^2 - y^2 = 1$ as

$$\left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 = 1,$$

$$(z+\bar{z})^2 + (z-\bar{z})^2 = 4,$$

$$z^2 + 2z\bar{z} + \bar{z}^2 + z^2 - 2z\bar{z} + \bar{z}^2 = 4,$$

$$2(z^2 + \bar{z}^2) = 4,$$

$$z^2 + \bar{z}^2 = 2.$$

9.2 (a) $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow |e^{i\theta}|^2 = \cos^2\theta + \sin^2\theta = 1 \Rightarrow |e^{i\theta}| = 1$

(b) $\overline{e^{i\theta}} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

9.9 $(1-z)(1+z+\dots+z^n)$

$$= \begin{array}{l} 1 + z + \dots + z^n \\ \quad - z - \dots - z^n - z^{n+1} \end{array}$$

$$= 1 - z^{n+1}.$$

Divide by $1-z$ (since $z \neq 1$) to get

$$1 + z + \dots + z^n = \frac{1-z^{n+1}}{1-z}.$$

[Note this is valid because $e^{i\theta} \neq 1$ for $0 < \theta < 2\pi$]

Now set $z = e^{i\theta}$ and take the real part. The left hand side becomes

$$1 + \cos\theta + \cos 2\theta + \dots + \cos n\theta.$$

The right hand becomes (here we have to be clever)

$$\mathrm{Re}\left(\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right) = \mathrm{Re}\left(\frac{e^{-i\theta/2} - e^{i(n+\frac{1}{2})\theta}}{e^{-i\theta/2} - e^{i\theta/2}}\right)$$

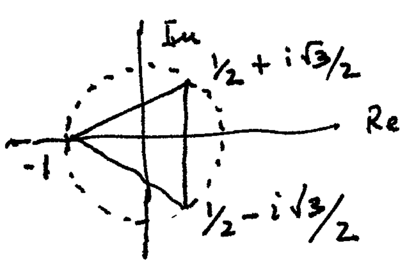
$$= \mathrm{Re}\left(\frac{e^{-i\theta/2} - e^{i(2n+1)\theta/2}}{-2i\sin\theta/2}\right)$$

$$= \frac{-1}{2\sin\theta/2}\,\mathrm{Im}\left(e^{-i\theta/2} - e^{i(2n+1)\theta/2}\right)$$

$$= \frac{1}{2\sin\theta/2}\left(\sin((2n+1)\theta/2) - \sin -\theta/2\right)$$

$$= \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2\sin\theta/2}.$$

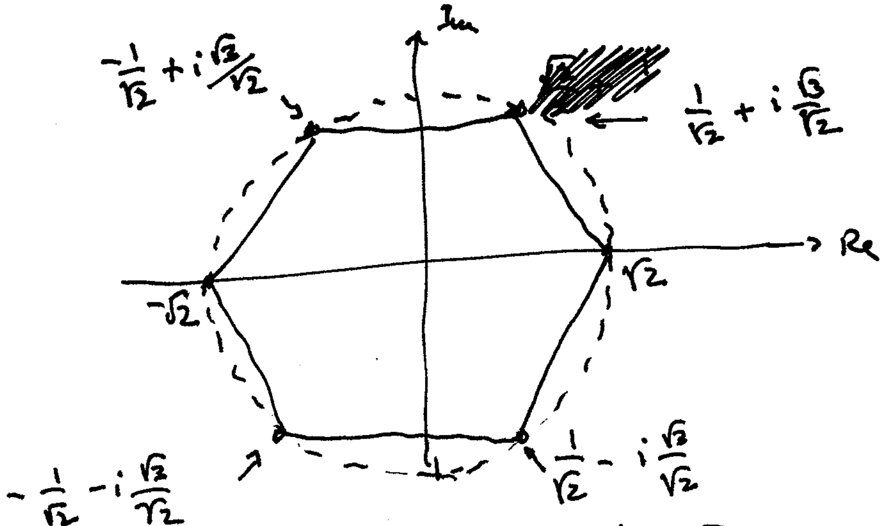
11.4 (a) $-1 = e^{i\pi}$, so the 3 values of $(-1)^{1/3}$ are $e^{i\pi/3}$, $e^{i3\pi/3} = e^{i\pi} = -1$, and $e^{i5\pi/3} = e^{-i\pi/3}$. In rectangular coordinates, $e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $e^{-i\pi/3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$. They are the vertices of a triangle:



The principal value of $\mathrm{Arg}(-1)$ is π, so $e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ is the principal root.

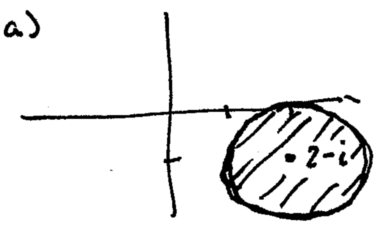
(b) $8 = 8e^{i\cdot 0}$ has 6^{th} roots $8^{1/6} = \underbrace{\sqrt[6]{8}}_{\text{(real)}} \cdot \{1, e^{i\pi/3}, e^{i2\pi/3}, e^{i\pi}, e^{i4\pi/3}, e^{i5\pi/3}\}$, or, since $\sqrt[6]{8} = \sqrt[6]{2^3} = \sqrt{2}$,

$\sqrt{2}\cdot\left\{\pm 1, \pm\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)\right\}$. They are the vertices of a hexagon on a circle of radius $\sqrt{2}$:



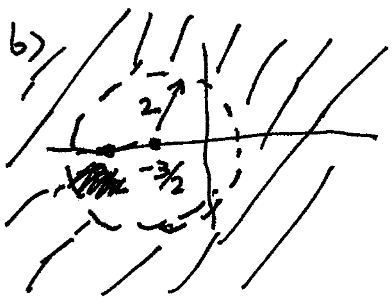
The principal root is the real root, $\sqrt{2}$.

12.1 a)



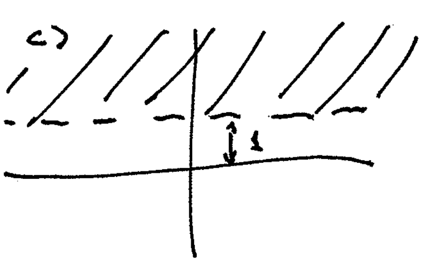
Not a domain because not open — this region includes its boundary.

b)



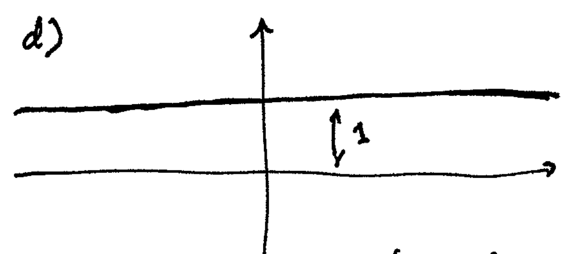
Exterior of the circle, excluding the boundary. It is a domain (but not a simply connected domain — we'll discuss this concept later).

c)

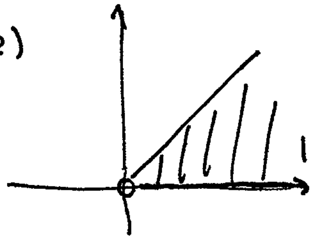


Is a domain

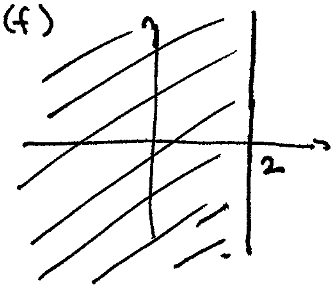
d)



Not open, so not a domain

(e) 

Includes the boundary rays (but not the origin), so not a domain

(f) 

Includes the boundary line, not a domain

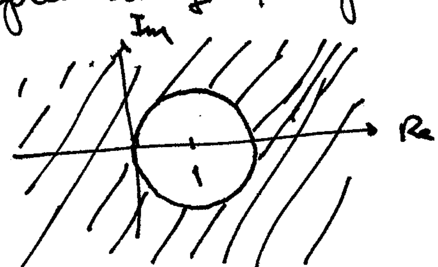
12.2 (a), (d), (f) are closed, (b), (c) are open, (e) is neither

12.3 Only (a).

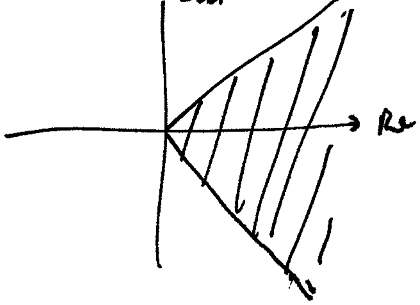
12.4 (a) The given set is the complement of the negative real axis. It's closure is the whole complex plane.

(b) The given set is the complement of the real axis. Again its closure is the whole plane

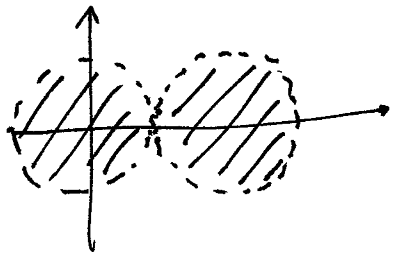
(c) If $z = x+iy$, then $\mathrm{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2}$, so $\mathrm{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$ is equivalent to $z \neq 0$ and $2x \le x^2+y^2$, or $1 \le (x-1)^2+y^2$, i.e. $|z-1| \ge 1$ and $z \neq 0$. Its closure includes $z=0$ and is the complement of the open disk of radius 1 centered at 1:



(d) z^2 is in the right half plane iff $-\pi/2 < \mathrm{Arg}\, z^2 < \pi/2$, $\iff -\pi/4 < \mathrm{Arg}\, z < \pi/4$. (and $z \neq 0$). The given set is the open sector between rays ~~at~~ angles $\pm \pi/4$, and its closure includes the boundary rays:



12.5 S is a union of two open disks:



It is not connected because there is no path in S from any point in one disk to a point in the other. (The point $z=1$ where the boundaries meet is not in S.)

12.6 By the definition of boundary point, every point z is either an interior point, an exterior point, or a boundary point of S. The exterior points are not in S, so every point of S is either an interior point or a boundary point. Hence the condition that no point of S is a boundary point (which your text uses as the definition of open set) is equivalent to every point of S being an interior point (the latter is usually taken to be the definition of open set, but the authors of your text probably think the definition in terms of boundary points is more intuitive).

14.3 $x^2-y^2-2y+i(2x-2xy) =$

$$\left(\frac{z+\bar z}{2}\right)^2 - \left(\frac{z-\bar z}{2i}\right)^2 - 2\,\frac{z-\bar z}{2i} + i\left(2\,\frac{z+\bar z}{2} - 2\,\frac{(z+\bar z)(z-\bar z)}{4i}\right)$$

$$= \frac{z^2+\bar z^2+2z\bar z}{4} + \frac{z^2+\bar z^2-2z\bar z}{4} - \frac{z-\bar z}{i} + i\left(z+\bar z - \frac{z^2-\bar z^2}{2i}\right)$$

$$= \frac{z^2+\bar z^2}{2} - \frac{z^2-\bar z^2}{2} + i\left(z+\bar z + z-\bar z\right) = \bar z^2 + 2iz.$$

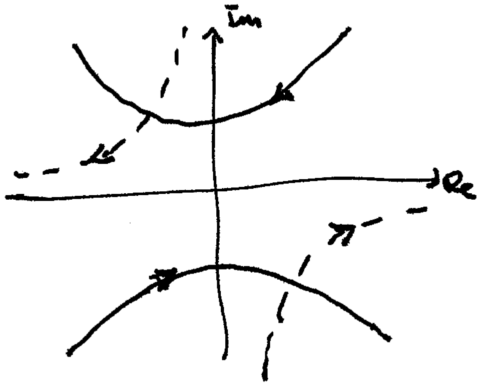
[Of course, since the answer was provided, you could substitute $z = x+iy$ into it and expand to get $x^2-y^2-2y+i(2x-2xy)$, but that's not really in the spirit of the problem.]

14.4 $z = re^{i\theta} = r\cos\theta + ir\sin\theta$

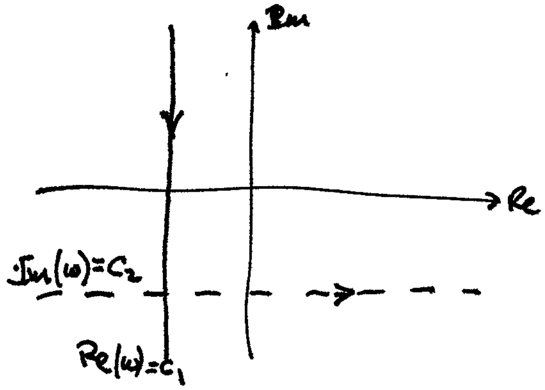
$\frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$

$f(z) = z+\frac{1}{z} = \left(r+\frac{1}{r}\right)\cos\theta + i\left(r-\frac{1}{r}\right)\sin\theta$

14.6 If $z = x+iy$, then $w = z^2 = x^2 - y^2 + 2ixy$. Hence $x^2-y^2 = c_1$ (<0) corresponds to the line $\mathrm{Re}(w) = c_1$ (<0) and $2xy = c_2$ (<0) to the line $\mathrm{Im}(w) = c_2$ (<0). Here is the picture:



z

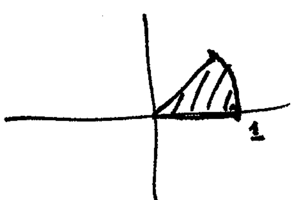


$w = z^2$

: $x^2-y^2=c_1$: $2xy = c_2$. Both branches of each hyperbola map to one line.

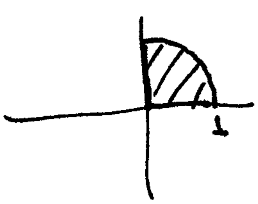
Since $\arg(w) = 2\arg(z)$ increases as $\arg(z)$ does, the orientations match with arrows in the direction of increasing argument.

14.8 Our region in the z plane looks like this:



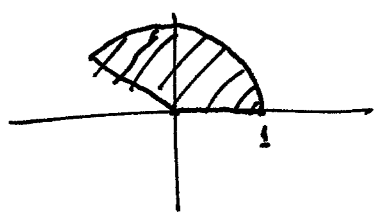
Both transformations (a) $w=z^2$ and (b) $w=z^3$ send points with $r = |z| \le 1$ to points with $|w| \le 1$, but (a) doubles the argument and (b) triples it, giving images in the w plane:

(a)



w

(b)



w

Additional problems:

1. (a) Let $\omega = e^{2\pi i/n}$. Then $z_k = \omega^k$, so

$$z_0 + z_1 + \dots + z_{n-1} = 1 + \omega + \omega^2 + \dots + \omega^{n-1} = \frac{1-\omega^n}{1-\omega} = 0$$

by Ex. 9.9 (note $\omega \neq 1$ for $n > 1$). since $\omega^n = 1$.

(b) The z_k are the vertices of a regular n-gon centered at 0, so their vector sum is 0 by symmetry.

(c) With $n=5$, $\omega = e^{2\pi i/5}$, we have $\bar{\omega} = \omega^{-1} = \omega^4$, so $x = \omega + \omega^4$, and $\bar{\omega}^2 = \omega^{-2} = \omega^3$, so $x^2 = \omega^2 + \bar{\omega}^2 + 2 = \omega^2 + \omega^3 + 2$.

Then $x^2 + x - 1 = \omega^2 + \omega^3 + 2 + \omega + \omega^4 - 1 = 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$, by (a).

(d) Let $x' = \omega^2 + \bar{\omega}^2 = \omega^2 + \omega^3$. Then $(x')^2 = \omega^4 + \omega^6 + 2\omega^5 = \underbrace{\bar{\omega} + \omega + 2}_{= \omega^4 + \omega + 2}$,

so $(x')^2 + x' - 1 = \omega^4 + \omega + 2 + \omega^2 + \omega^3 - 1 = 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

(e) $\cos(\pi/5) = -\cos(\pi - \pi/5) = -\cos(4\pi/5) = -\mathrm{Re}\, e^{4\pi i/5} = -\mathrm{Re}\, \omega^2$

$= -\frac{\omega^2 + \bar{\omega}^2}{2} = -x'/2$.

$\cos(2\pi/5) = \mathrm{Re}\, e^{2\pi i/5} = \mathrm{Re}\, \omega = \frac{\omega + \bar{\omega}}{2} = x/2$.

Now x, x' are given by $\frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$, with $x = 2\cos(2\pi/5) > 0$ and $x' = -2\cos(\pi/5) < 0$. Hence

$$\cos(\pi/5) = \frac{1+\sqrt{5}}{4}, \qquad \cos(2\pi/5) = \frac{\sqrt{5}-1}{4}$$

2. The answer to 14.1 (b) should be $z \neq 0$. The answer $\mathrm{Re}\, z \neq 0$ goes with 14.1 (c), since $z + \bar{z} = 2\,\mathrm{Re}(z)$.