PS1 solutions

2.2 If $z = x+iy$, then $iz = -y+ix$. Hence
$\mathrm{Re}(iz) = -y = -\mathrm{Im}(z)$, $\mathrm{Im}(iz) = x = \mathrm{Re}(z)$.

2.6(b). Let $z = x+iy$, $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$.

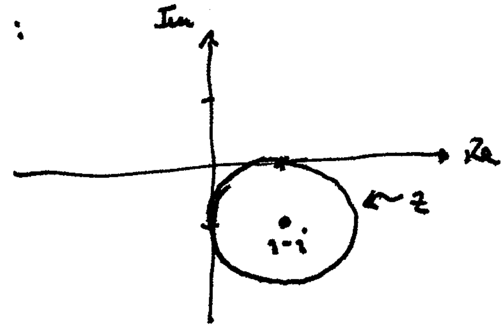
Then
$$z(z_1+z_2) = (x+iy)(x_1+x_2+i(y_1+y_2))$$
$$= x(x_1+x_2) - y(y_1+y_2) + i\left(x(y_1+y_2)+y(x_1+x_2)\right)$$
$$= xx_1+xx_2-yy_1 \cancel{+} -yy_2 + i(xy_1+xy_2+yx_1+yx_2).$$

Compare this with
$$zz_1+zz_2 = (x+iy)(x_1+iy_1) + (x+iy)(x_2+iy_2)$$
$$= xx_1-yy_1+i(xy_1+yx_1) + xx_2-yy_2+i(xy_2+yx_2)$$
$$= xx_1+xx_2-yy_1-yy_2 + i(xy_1+xy_2+yx_1+yx_2).$$

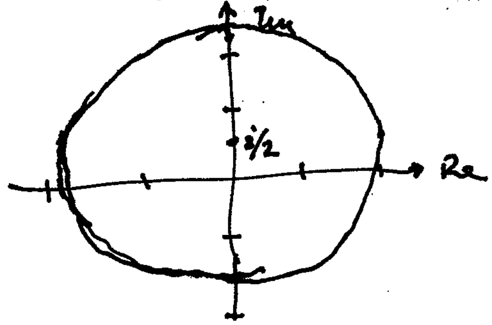
2.11 With $z=x+iy$, the equation $z^2+z+1=0$ becomes
$$x^2-y^2+2ixy+x+iy+1=0$$
$$x^2+x+1-y^2+i(2xy+y)=0.$$
The solution must have $2xy+y=0$ and $x^2+x+1-y^2=0$. As in the hint, since $z^2+z+1=0$ has no real roots, we cannot have $y=0$. So $2xy+y=0 \Rightarrow (2x+1)y=0 \Rightarrow 2x+1=0 \Rightarrow x=-\frac{1}{2}$. Then $x^2+x+1=y^2$ gives $y^2=\frac{3}{4}$, $y=\pm\frac{\sqrt{3}}{2}$.

So there are two solutions: $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

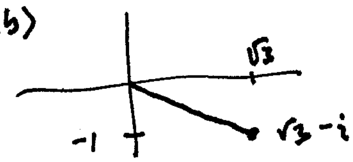
5.5(a) The condition means z is on a circle of radius 1 centered at $1-i$:



6.2 (b) Since $|\bar{w}| = |w|$, $|2\bar{z}+i| = 4$ is equivalent to $|2z - i| = 4$, i.e. $2z$ is on a circle of radius 4 centered at i, or z on circle of radius 2 centered at $i/2$.



9.1 (b)



$\sqrt{3} - i$ has argument $-\pi/6$, since it's in the IVth quadrant, and $\tan(-\pi/6) = -1/\sqrt{3}$.

Therefore $6 \cdot (-\pi/6) = -\pi$ is a value of $\arg(\sqrt{3}-i)^6$, i.e. $(\sqrt{3}-i)^6$ is a negative real number. The principal value of the argument is $\operatorname{Arg}(\sqrt{3}-i)^6 = \pi$.

[9.5 (d) is below]

11.6 Multiply $z_0 = \sqrt{2}\, e^{i\pi/4} = 1+i$ by the four 4^{th} roots of 1, namely $1, i, -1, -i$, to find all four values of $(-4)^{1/4}$:

$1+i, \ -1+i, \ -1-i, \ 1-i.$

In conjugate pairs, these must be the roots of the quadratic factors of z^4+4. Thus one of the factors, with roots $1 \pm i$, is

$$(z-1-i)(z-1+i) = z^2 - 2z + 2,$$

and the other, with roots $-1 \pm i$, is

$$(z+1-i)(z+1+i) = z^2 + 2z + 2,$$

giving $z^4 + 4 = (z^2 - 2z + 2)(z^2 - 2z + 2)$.

9.5 (d) Since $1 + \sqrt{3}\, i = 2e^{i\pi/3}$, $(1+\sqrt{3}i)^{-10} = 2^{-10} e^{-i\,10\pi/3}$

$$= 2^{-10} e^{i\,2\pi/3} = 2^{-10}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2^{-11}(-1 + i\sqrt{3})$$

11.8 a) The usual derivation of the quadratic formula works for complex numbers too: complete the square, writing

$$az^2 + bz + c = 0$$

as

$$a\left(z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2\right) - \frac{b^2}{4a} + c = 0,$$

or

$$a\left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c = \frac{b^2 - 4ac}{4a}$$

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$z + b/2a = (b^2 - 4ac)^{1/2}/2a$$ (both values of $(b^2-4ac)^{1/2}$ to be considered)

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a}.$$

b) Solving $z^2 + 2z + (1-i) = 0$, with $a=1$, $b=2$, $c = 1-i$, $b^2 - 4ac = 4 - (4-4i) = 4i = 4e^{i\pi/2}$. One value of $(b^2-4ac)^{1/2}$ is then $2e^{i\pi/4} = 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2} + \sqrt{2}i$, and the other is -1 time this, or $-\sqrt{2} - \sqrt{2}i$. This gives the two solutions

$$z = \frac{-2 \pm (\sqrt{2} + \sqrt{2}i)}{2},$$

that is, $-1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

Additional probem

(a) $2 + \sqrt{3} + i$ (b) $-1 - 2\sqrt{3} + (-2 + \sqrt{3})i$ (c) $-\frac{1}{4} - \frac{\sqrt{3}}{4}i$

(d) $-\frac{1}{4} + \frac{\sqrt{3}}{2} + \left(-\frac{1}{2} - \frac{\sqrt{3}}{4}\right)i$ (e) $|w| = \sqrt{5}$, $|z| = 2$, $|wz| = 2\sqrt{5}$, $|w+z| = 2+\sqrt{3}$ (note $< |w| + |z|$ by Δ inequality, but not $= |w| + |z|$)

(f) $\text{Arg}(w) = \tan^{-1}(2)$ (specifically, the value of $\theta = \tan^{-1}(2)$ in $(0, \pi/2)$, which is approximately $\theta \approx 1.10715$), $\text{Arg}(z) = \frac{2\pi}{3}$, $\text{Arg}(wz) = \frac{2\pi}{3} + \tan^{-1}(2) - 2\pi$ ($\frac{2\pi}{3} + \tan^{-1}(2) = \text{Arg}(z) + \text{Arg}(w)$ is a value of $\arg(wz)$, but not the principal value)

(g) $-16 - 16\sqrt{3}i$ (h) $w\bar{w} = |w|^2 = 5$ (i) $wz + \overline{wz} = wz + \bar{w}\bar{z} = 2\,\text{Re}(wz) = -1 - 2\sqrt{3}$.