Math 185—Introduction to Complex Analysis
Haiman, Summer 2014

Midterm Exam

You may consult the class textbook and your own notes. No other printed materials or eletronic devices may be used.

1. [12 pts] Show that if $\mathrm{Re}(z) > 0$, then

$$\mathrm{Arg}(z) = \frac{\mathrm{Log}(z/\overline{z})}{2i}.$$

2. (a) [9 pts] Using the identity $z^3 + 1 = (z+1)(z^2 - z + 1)$, show that the roots of the quadratic equation $z^2 - z + 1 = 0$ are the two complex cube roots of -1 which are not real.

(b) [6 pts] Derive the exact values of $\cos(\pi/3)$ and $\sin(\pi/3)$ from part (a).

3. [14 pts] Without explicitly calculating the derivatives, show that the function $u(x, y) = \ln(x^2 + y^2)$ is harmonic, *i.e.*, it satisfies the Laplace equation $u_{xx} + u_{yy} = 0$, for all $(x, y) \neq (0, 0)$.

4. (a) [8 pts] Show that if a is a non-zero complex number, and one value of a^i is real, then every value of a^i is real.

(b) [5 pts] For which complex numbers $a \neq 0$ are the values of a^i real?

5. (a) [10 pts] Prove that if a function $f(z)$ is analytic on a simply connected domain D, then f has an antiderivative $F(z)$ on D.

(b)[6 pts] Give an example showing that part (a) does not always hold if the domain D is not assumed to be simply connected.

6. [14 pts] Show that the function

$$f(z) = e^{-2xy} \cos\left(x^2 - y^2\right) + ie^{-2xy} \sin\left(x^2 - y^2\right),$$

where $z = x + iy$, is entire.

7. [16 pts] Given $R > 1$, let S be the region defined by $|z| \leq R$, $\mathrm{Re}(z) \geq 0$ and $\mathrm{Im}(z) \geq 1$. Let C be the boundary of S. By evaluating the contour integral

$$\int_C \frac{1}{z^2}\, dz$$

and letting R approach infinity, find

$$\int_0^\infty \frac{x^2 - 1}{(x^2 + 1)^2}\, dx.$$