Midterm Exam Solutions

1. If $Re(z) > 0$, then $z = re^{i\theta}$ with $r \neq 0$ and $-\pi/2 < \theta < \pi/2$.
Hence $z/\bar{z} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{2i\theta}$, and $-\pi < 2\theta < \pi$. The principal value $\mathrm{Log}(z/\bar{z})$ is therefore $\ln(1) + 2i\theta = 2i\theta$, so $\frac{\mathrm{Log}(z/\bar{z})}{2i} = \theta$, which is also the principal value $\mathrm{Arg}(z)$.

2. (a) The cube roots of -1 are the roots of $z^3+1=0$. The real one is $z=-1$. The others have $z+1 \neq 0$, so $(z+1)(z^2-z+1)=0 \Rightarrow$ $z^2-z+1=0$.
(b) Since $-1 = e^{i\pi}$, its cube roots are $e^{i\pi/3}$, $e^{i3\pi/3}$, $e^{i5\pi/3} = e^{-i\pi/3}$ of these, $e^{i3\pi/3} = -1$, and the others, $e^{\pm i\pi/3} = \cos\frac{\pi}{3} \pm i\sin\pi/3$, are the roots of $z^2-z+1=0$. By the quadratic formula,

$$\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

This gives $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ (since $\sin\pi/3 > 0$).

3. $\ln(x^2+y^2) = 2\ln(\sqrt{x^2+y^2}) = 2\ln r$ is the real part of any branch of $2\log(z)$. Since $\log(z)$ has a branch analytic in a neighborhood of z for every $z \neq 0$, $\ln(x^2+y^2)$ is harmonic for all $(x,y) \neq (0,0)$.

4. a) If $a = re^{i\theta}$ $(r \neq 0)$ then all values of a^i are

$$e^{i\log a} = e^{i(\ln r + i(\theta + 2n\pi))} = e^{-2n\pi}\left(e^{-\theta}(\cos\ln r + i\sin\ln r)\right).$$

Since these values differ by factors of $e^{-2n\pi}$, if one is real, they all are.
b) The condition for a^i to be real is $\sin(\ln r) = 0$, that is, $\ln r = k\pi$, or $r = e^{k\pi}$. So a^i is real if and only if $|a| = e^{k\pi}$ for k an integer.

5. a) If f is analytic on D, then $\int_C f(z)\,dz = 0$ for every closed contour C in D by Cauchy-Goursat (since D is simply connected). By the theorem in §28, this is equivalent to f having an antiderivative on D.

b) Take $D = \mathbb{C} \setminus \{0\}$ and $f(z) = 1/z$. If C is the circle $|z|=1$, then C is in D and $\int_C \frac{1}{z}\,dz = 2\pi i \neq 0$, so $f(z) = \frac{1}{z}$ cannot have an antiderivative (i.e. no branch of $\log(z)$ is analytic on all of D).

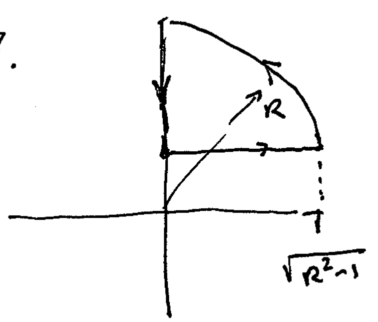
6. Either verify the Cauchy-Riemann equations:

$$u_x = -2y e^{-2xy} \cos(x^2-y^2) - 2x e^{-2xy} \sin(x^2-y^2) = v_y$$

$$u_y = -2x e^{-2xy} \cos(x^2-y^2) + 2y e^{-2xy} \sin(x^2-y^2) = -v_x,$$

or observe that $f(z) = e^{iz^2}$:

$$e^{iz^2} = e^{i(x+iy)^2} = e^{i((x^2-y^2)+2ixy)} = e^{-2xy}\left(\cos(x^2-y^2) + i\sin(x^2-y^2)\right).$$

7. 

$$\int_C \frac{1}{z^2} = \int_0^{\sqrt{R^2-1}} \frac{1}{(x+i)^2}\,dx + \int_{\sin^{-1}\frac{1}{R}}^{\pi/2} \frac{1}{R^2 e^{2i\theta}} \cdot iRe^{i\theta}\,d\theta$$

$$+ \int_R^1 \frac{1}{(iy)^2}\, i\,dy = 0 \quad \text{by Cauchy-Goursat.}$$

The last integral is pure imaginary, and $\mathrm{Re}\left(\frac{1}{(x+i)^2}\right) = \frac{x^2-1}{(x^2+1)^2}$,

so $\mathrm{Re}\int_0^{\sqrt{R^2-1}} \frac{1}{(x+i)^2}\,dx \longrightarrow \int_0^\infty \frac{x^2-1}{(x^2+1)^2}\,dx$ as $R\to\infty$.

The middle integral has

$$\left|\int_{\sin^{-1}1/R}^{\pi/2} \frac{i\,d\theta}{Re^{i\theta}}\right| \le \int_{\sin^{-1}1/R}^{\pi/2} \left|\frac{i}{Re^{i\theta}}\right| d\theta = \int_{\sin^{-1}1/R}^{\pi/2} \frac{1}{R}\,d\theta \le \int_0^{\pi/2} \frac{1}{R}\,d\theta \le \pi/2R \to 0$$

as $R\to\infty$. Taking real parts and letting $R\to\infty$ therefore gives

$$\int_0^\infty \frac{x^2-1}{(x^2+1)^2}\,dx = 0$$