# Notes on Gamma and Zeta

Mark Haiman, Aug. 2014

## 1. The gamma function

Initially, we define the *gamma function* by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx \qquad (\mathrm{Re}(z) > 0). \tag{1}$$

If $z$ is real, the improper integral converges at the upper end because $e^{-x}$ goes to zero much faster than the growth of any power $x^{z-1}$. This convergence is uniform on $z \leq b$ because if $z \leq b$, then $x^{z-1} \leq x^{b-1}$ for $x > 1$. It converges at the lower end for $z > 0$, uniformly on $z \geq a$, for any $a > 0$, because if $z \geq a$ then $x^{z-1} \leq x^{a-1}$ for $x < 1$.

For complex $z$, since $|x^{z-1}| = x^{\mathrm{Re}(z)-1}$, it follows that the integral converges absolutely for $\mathrm{Re}(z) > 0$, and uniformly on $a \leq \mathrm{Re}(z) \leq b$, for any $0 < a < b$. In particular, if $\mathrm{Re}(z) > 0$, the integral converges uniformly on a neighborhood of $z$, so we can differentiate under the integral sign to get

$$\Gamma'(z) = \int_0^\infty x^{z-1} e^{-x} \ln(x)\, dx.$$

This proves that $\Gamma(z)$ is analytic on $\mathrm{Re}(z) > 0$.

Integrating by parts with $u = x^z$ and $v = -e^{-x}$ gives

$$\int_0^\infty x^z e^{-x}\, dx = -x^z e^{-x}\Big|_0^\infty + z \int_0^\infty x^{z-1} e^{-x}\, dx \qquad (\mathrm{Re}(z) > 0).$$

The first term on the right hand side vanishes, yielding the identity

$$\Gamma(z+1) = z\Gamma(z). \tag{2}$$

In particular, since $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$, it follows that

$$\Gamma(n+1) = n! \tag{3}$$

for every integer $n \geq 0$, so we can think of $\Gamma(z+1)$ as a kind of continuous version of the factorial function. Another special value is

$$\Gamma(1/2) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx = \sqrt{\pi}, \tag{4}$$

which can be derived from the classical integral

$$\int_0^\infty e^{-u^2}\, du = \sqrt{\pi}/2 \tag{5}$$

by letting $u = \sqrt{x}$.

We can rewrite (2) as

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \tag{6}$$

and use this to define an analytic continuation of $\Gamma(z)$ to the domain $D_1 = \{\mathrm{Re}(z) > -1\} - \{0\}$, then to $D_2 = \{\mathrm{Re}(z) > -2\} - \{0, -1\}$, and so on in succession, the extension to each $D_n = \{\mathrm{Re}(z) > -n\} - \{0, -1, \ldots, 1-n\}$ being consistent with the previous one because (2) already holds on $D_{n-1}$. This proves:

PROPOSITION 1. *The gamma function $\Gamma(z)$, as defined by* (1), *has an analytic continuation (necessarily unique) to the domain $\mathbb{C} - \{0, -1, -2, \ldots\}$, and satisfies* (2) *for all $z$ in this domain.*

From here on, $\Gamma(z)$ will stand for the gamma function analytically continued to this larger domain.

## 2. The beta function

To understand more about the gamma function it will be helpful to introduce its cousin, the *beta function*, defined by

$$B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx \qquad (\mathrm{Re}(r), \mathrm{Re}(s) > 0). \tag{7}$$

Like (1), this integral converges absolutely and uniformly on a neighborhood of any $(r,s)$ such that $\mathrm{Re}(r), \mathrm{Re}(s) > 0$ and defines a function analytic in each variable. By making the change of variable $y = 1 - x$ in (7) it is easy to verify that

$$B(r,s) = B(s,r). \tag{8}$$

The relationship between $\Gamma(z)$ and $B(r,s)$ comes about by making the change of variable $u = x + y$ in

$$\Gamma(r)\Gamma(s) = \left(\int_0^\infty x^{r-1}e^{-x}\,dx\right)\left(\int_0^\infty y^{s-1}e^{-y}\,dy\right) = \int_0^\infty \int_0^\infty x^{r-1}y^{s-1}e^{-(x+y)}\,dx\,dy$$

to get

$$\Gamma(r)\Gamma(s) = \int_0^\infty e^{-u}\left(\int_0^u x^{r-1}(u-x)^{s-1}\,dx\right)du. \tag{9}$$

Letting $x = uy$ in the inner integral, we see that

$$\int_0^u x^{r-1}(u-x)^{s-1}\,dx = u^{r+s-1}\int_0^1 y^{r-1}(1-y)^{s-1}\,dy = u^{r+s-1}B(r,s).$$

The right hand side of (9) is therefore equal to

$$B(r,s)\int_0^\infty u^{r+s-1}e^{-u}\,du = B(r,s)\Gamma(r+s),$$

yielding the identity

$$\Gamma(r)\Gamma(s) = B(r,s)\Gamma(r+s). \tag{10}$$

We will see shortly that $\Gamma(z)$ has simple poles at $z = 0, -1, -2, \ldots$, and no zeroes. Temporarily taking this for granted, it is possible to continue $B(r,s)$ analytically to all $r, s \notin \{0, -1, -2, \ldots\}$ by rewriting (10) as

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}. \tag{11}$$

In these notes, however, we will only need to make use of the beta function on the domain where it is defined by (7).

We now make a change of variable

$$x = \frac{u}{u+1}, \qquad 1 - x = \frac{1}{u+1}, \qquad dx = \frac{du}{(u+1)^2}$$

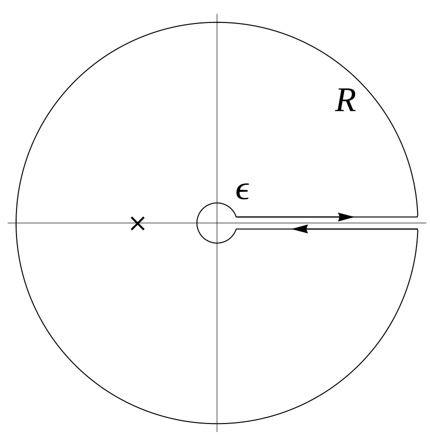
in (7), obtaining an alternative formula

$$B(r,s) = \int_0^\infty \frac{u^{r-1}}{(u+1)^{r+s}} du \qquad (\mathrm{Re}(r), \mathrm{Re}(s) > 0). \tag{12}$$

One advantage of this formula is that for $r + s = 1$, we can evaluate it using the residue theorem. To do this we consider the contour integral

$$\int_C \frac{(-z)^{r-1}}{z+1} \, dz, \tag{13}$$

where we use the principal branch $(-z)^{r-1} = e^{(r-1)\,\mathrm{Log}(-z)}$ and the contour shown here.




Since $\mathrm{Log}(z)$ has a branch cut on the negative real axis, $(-z)^{r-1}$ has a branch cut on positive real axis. Along the upper side of the cut, we have $\mathrm{Arg}(-z) = -\pi$ and $(-z)^{r-1} = e^{-i\pi(r-1)} x^{r-1}$. Along the lower side, we have $\mathrm{Arg}(-z) = \pi$ and $(-z)^{r-1} e^{i\pi(r-1)}$.

As $R \to \infty$ and $\epsilon \to 0$, the contribution to our contour integral from the legs on each side of the real axis is therefore

$$(e^{-i\pi(r-1)} - e^{i\pi(r-1)}) \int_0^\infty \frac{x^{r-1}}{x+1}\, dx = -2i\sin(\pi(r-1))B(r, 1-r).$$

The integral on the big circle $|z| = R$ goes to zero as $R \to \infty$, because $R\,|z^{r-1}/(z+1)| \sim R\,R^{\rho-1}/R = R^{\rho-1}$, where $\rho = \mathrm{Re}(r) = 1 - \mathrm{Re}(s) < 1$. Similarly, the integral on the small circle $|z| = \epsilon$ goes to zero as $\epsilon \to 0$, because $\epsilon\,|z^{r-1}/(z+1)| \sim \epsilon\,\epsilon^{\rho-1} = \epsilon^\rho$ there, and $\rho > 0$. Hence

$$\int_C \frac{(-z)^{r-1}}{z+1} dz = -2i\sin(\pi(r-1))B(r, 1-r).$$

Since the function $(-z)^{r-1}/(z+1)$ has a pole at $z = -1$ with residue $1^{r-1} = 1$, the residue theorem gives

$$-2i\sin(\pi(r-1))B(r, 1-r) = 2\pi i,$$

or

$$\frac{\sin(\pi(1-r))}{\pi} B(r, 1-r) = 1.$$

Using (11) with $\Gamma(1) = 1$ and the identity $\sin(\pi - x) = \sin(x)$, this becomes

$$\Gamma(z)\Gamma(1-z)\frac{\sin \pi z}{\pi} = 1 \tag{14}$$

In deriving (14), we assumed that $0 < \mathrm{Re}(z) < 1$. However, since this is an identity between analytic functions, it follows that it holds throughout their domains of definition, that is, for every complex number $z$ which is not an integer.

One useful consequence of (14) is the following.

Proposition 2. *The gamma function $\Gamma(z)$ has no zeroes, and has a simple pole of order $(-1)^n/n!$ at $z = -n$, for every integer $n \geq 0$.*

To prove the proposition, note that (14) implies that $\Gamma(z)$ has no zeroes at non-integer values of $z$. Since $\Gamma(n) = (n-1)!$ for positive integers $n$, it has no zeroes in its domain. For integers $n \geq 0$, the function $\sin(\pi z)/\pi$ has a zero at $z = -n$ with first derivative $\cos(-\pi n) = (-1)^n$. Using $\Gamma(1 - (-n)) = n!$, it follows from (14) that $\Gamma(z)$ has simple pole at $z = -n$ with residue $(-1)^n/n!$. □

As another nice application, we can derive the value $\Gamma(1/2) = \sqrt{\pi}$ in (4) in a new way, and therefore also deduce (5), by setting $z = 1/2$ in (14).

## 3. The product formula

By iterating (6), we get

$$\Gamma(z) = \frac{1}{z}\frac{1}{z+1}\cdots\frac{1}{z+n-1}\Gamma(z+n). \tag{15}$$

If $\Gamma(z+n)$ were to approach a limit as $n \to \infty$, this would give an infinite product formula for $\Gamma(z)$. Of course, $\lim_{n\to\infty} \Gamma(z+n)$ does not exist, so the corresponding product doesn't converge, but we can try to improve it. For instance, multiplying and dividing by $(n-1)! = \Gamma(n)$, we can rewrite (15) as

$$\Gamma(z) = \frac{1}{z} \left( \prod_{k=1}^{n-1} \frac{1}{1+z/k} \right) \frac{\Gamma(z+n)}{\Gamma(n)}.$$

As it turns out, $\lim_{n\to\infty} \Gamma(z+n)/\Gamma(n)$ does not exist either, so this still does not lead to a convergent infinite product. However, we can fix this with the aid of the following limit formula.

Proposition 3. For all $z$ in the domain of the gamma function,

$$\lim_{n\to\infty} \frac{n^{-z}\Gamma(z+n)}{\Gamma(n)} = 1. \tag{16}$$

To prove the proposition, we first observe that (2) implies

$$\frac{n^{-z}\Gamma(z+n)}{\Gamma(n)} = \left( \frac{z-1+n}{n} \right) \frac{n^{1-z}\Gamma(z-1+n)}{\Gamma(n)},$$

and therefore (16) holds for $z$ if and only if it holds for $z-1$, since the factor in parentheses has limit equal to 1. Using this, we can reduce to the case $\text{Re}(z) < 0$, or equivalently to proving that

$$\lim_{n\to\infty} \frac{n^{z}\Gamma(n-z)}{\Gamma(n)} = 1 \tag{17}$$

for $\text{Re}(z) > 0$. Now, using (10) and (12), since $\Gamma(z)$ has no zeroes, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^{z}\Gamma(n-z)}{\Gamma(n)} &= \frac{1}{\Gamma(z)} \lim_{n\to\infty} n^z B(z, n-z) \\
&= \frac{1}{\Gamma(z)} \lim_{n\to\infty} n^z \int_0^\infty \frac{x^{z-1}}{(1+x)^n}\, dx \\
&= \frac{1}{\Gamma(z)} \lim_{n\to\infty} \int_0^\infty \frac{u^{z-1}}{(1+u/n)^n}\, du,
\end{aligned}$$

where we got the last integral by substituting $u = nx$. Recalling that

$$\lim_{n\to\infty} (1+u/n)^{-n} = e^{-u},$$

we see that the last integral becomes $\Gamma(z)$, giving (17), provided it is permissible to take the limit inside the integral. This can be justified using the fact that the improper integral is absolutely convergent for $\text{Re}(z) > 0$, and that $(1+u/n)^{-n} = e^{-n\ln(1+u/n)}$ converges to $e^{-u}$ monotonically, because $-n\ln(1+u/n)$ is a decreasing function of $n$. $\square$

Recall that the limit

$$\gamma = \lim_{n\to\infty}\left((1+\frac{1}{2}+\cdots+\frac{1}{n-1}) - \ln(n)\right)$$

exists; its value $\gamma \approx 0.577216$ is known as *Euler's constant.*

Together with Proposition 3, this suggests that we adjust (15) by writing it as follows:

$$\Gamma(z) = e^{\left(\ln(n)-(1+\frac{1}{2}+\cdots+\frac{1}{n-1})\right)z}\frac{1}{z}\left(\prod_{k=1}^{n-1}\frac{e^{z/k}}{1+z/k}\right)\frac{n^{-z}\Gamma(z+n)}{\Gamma(n)}. \tag{18}$$

Taking the limit as $n \to \infty$ in (18) now yields the following result.

PROPOSITION 4. *The gamma function has a convergent infinite product representation*

$$\Gamma(z) = e^{-\gamma z}\frac{1}{z}\prod_{k=1}^{\infty}\frac{e^{z/k}}{1+z/k}, \tag{19}$$

*where $\gamma$ is Euler's constant.*

Note that (19) makes sense for all $z$ in the domain of $\Gamma(z)$, and that each factor in the denominator contributes a pole at one of the values $z = 0, -1, -2, \ldots$.

As an interesting application, we can combine (19) with (14) to obtain infinite product representations

$$\frac{\pi}{\sin \pi z} = \Gamma(z)\Gamma(1-z) = -z\Gamma(z)\Gamma(-z) = \frac{1}{z}\prod_{k=1}^{\infty}\frac{1}{1-z^2/k^2}$$

$$\sin \pi z = \pi z\prod_{k=1}^{\infty}(1-z^2/k^2).$$

Taking the logarithm and differentiating gives a nice infinite series representation

$$\pi \cot \pi z = \frac{1}{z} + \sum_{k=1}^{\infty}\frac{2z}{z^2-k^2} = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{z-k}+\frac{1}{z+k}\right).$$

Here we see that even though the series

$$\sum_{k=-\infty}^{\infty}\frac{1}{z-k}$$

is divergent when summed separately at each end, it converges beautifully to $\pi \cot \pi z$ when the terms are grouped in a sufficiently symmetrical manner.

## 4. The Gauss-Legendre duplication formula

Although $\log\Gamma(z)$ is multiple-valued, its derivative $(\log\Gamma(z))' = \Gamma'(z)/\Gamma(z)$ makes sense as a single-valued function, analytic on the whole domain of $\Gamma(z)$ since $\Gamma(z)$ has no zeroes. From the product formula (19), we obtain

$$(\log\Gamma(z))' = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{z+k}\right), \tag{20}$$

$$(\log\Gamma(z))'' = \sum_{k=0}^{\infty}\frac{1}{(z+k)^2}. \tag{21}$$

Evaluating (20) when $z$ is a positive integer $n$ gives a nice expression

$$\frac{\Gamma'(n)}{(n-1)!} = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k},$$

but here we will instead concern ourselves chiefly with (21).

Consider the product

$$G_n(z) = \prod_{j=0}^{n-1}\Gamma(\frac{z+j}{n}) = \Gamma(\frac{z}{n})\Gamma(\frac{z+1}{n})\cdots\Gamma(\frac{z+n-1}{n}).$$

Using (21), we find

$$(\log G_n(z))'' = \sum_{j=0}^{n-1}\frac{1}{n^2}\sum_{k=0}^{\infty}\frac{1}{((z+j)/n+k)^2} = \sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\frac{1}{(z+j+kn)^2}.$$

Since every non-negative integer $m$ has a unique expression as $m = kn + j$ for integers $k \geq 0$ and $0 \leq j \leq n-1$, the last sum is equal to the right hand side of (21). Thus we have

$$(\log G_n(z))'' = (\log\Gamma_n(z))'',$$

or

$$(\log G_n(z)/\Gamma(z))'' = 0.$$

This implies that $G_n(z)/\Gamma(z)$ is the exponential of a linear function, that is,

$$G_n(z) = e^{A+Bz}\Gamma(z) = C\,M^z\,\Gamma(z)$$

for constants $C$ and $M$ (depending on $n$). Since $G_n(z)$ and $\Gamma(z)$ are real and positive for real $z > 0$, the constants $C$ and $M$ are real and postive. We shall now evaluate them.

Using (2), we get

$$G_n(z+n) = \frac{z}{n}\,\frac{z+1}{n}\cdots\frac{z+n-1}{n}\,G(z).$$

Combining this with

$$\Gamma(z+n) = z(z+1)\cdots(z+n-1)\Gamma(z),$$

we discover that

$$\frac{n^{z+n}G_n(z+n)}{\Gamma(z+n)} = \frac{n^z G_n(z)}{\Gamma(z)}.$$

In other words, the expression $n^z G_n(z)/\Gamma(z) = CM^z n^z$ is a periodic function of $z$ with period $n$. Since $C$ and $M$ are real, this is only possible if $M = 1/n$. We now have

$$G_n(z) = C\, n^{-z}\, \Gamma(z).$$

At $z = 0$, $\Gamma(z)$ has a pole with residue 1, and the factor $\Gamma(z/n)$ in $G_n(z)$ therefore has a pole with residue $n$. Cancelling these factors and setting $z = 0$ gives

$$C = n\Gamma(\frac{1}{n})\cdots\Gamma(\frac{n-1}{n}) = n\left(\prod_{j=1}^{n-1}\Gamma(\frac{j}{n})\Gamma(\frac{n-j}{n})\right)^{1/2} = n\,\frac{\pi^{(n-1)/2}}{\left(\prod_{j=1}^{n-1}\sin(j\pi/n)\right)^{1/2}},$$

by (14). Now we invoke the trigonometric identity

$$\sin(n\theta) = 2^{n-1}\prod_{j=0}^{n-1}\sin(\theta + j\pi/n), \tag{22}$$

a generalization of the more familiar identity

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\sin(\theta)\sin(\theta + \pi/2)$$

for $n = 2$. Accepting (22) for the moment, we have

$$\frac{\sin(n\theta)}{\sin\theta} = 2^{n-1}\prod_{j=1}^{n-1}\sin(\theta + j\pi/n).$$

Letting $\theta$ approach zero then gives $\prod_{j=1}^{n-1}\sin(j\pi/n) = n/2^{n-1}$, whence $C = n^{1/2}(2\pi)^{(n-1)/2}$. Putting all this together gives *Gauss's identity*

$$\Gamma(\frac{z}{n})\Gamma(\frac{z+1}{n})\cdots\Gamma(\frac{z+n-1}{n}) = (2\pi)^{(n-1)/2}n^{\frac{1}{2}-z}\Gamma(z). \tag{23}$$

The special case $n = 2$, known as *Legendre's duplication formula*, reads

$$\Gamma(\frac{z}{2})\Gamma(\frac{z+1}{2}) = 2^{1-z}\sqrt{\pi}\,\Gamma(z). \tag{24}$$

Note that setting $z = 1$ in this formula gives $\Gamma(1/2) = \sqrt{\pi}$ once again.

Now we briefly turn our attention to the derivation of (22). Setting $\omega = e^{i\pi/n}$ and $z = e^{i\theta}$, (22) reads

$$z^n - z^{-n} = i^{1-n}\prod_{j=0}^{n-1}(z\,\omega^j - z^{-1}\omega^{-j}).$$

This can be derived algebraically by substituting $z^2$ for $z$ in the factorization

$$z^n - 1 = \prod_{j=0}^{n-1} (z - \omega^{2n}), \tag{25}$$

which express the fact that the powers of $\omega^2$ are the $n$-th roots of unity. One also needs to know that $\omega^0 \omega^1 \cdots \omega^{n-1} = i^{n-1}$. Setting $z = 0$ in (25) gives the square of this equation, showing that $\omega^0 \omega^1 \cdots \omega^{n-1} = \pm i^{n-1}$. To establish the sign we need only note that both sides of (22) are positive for $0 < \theta < \pi/n$.

## 5. The Riemann zeta function

The *Riemann zeta function* is initially defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\mathrm{Re}(s) > 1).$$

It is traditional to use the variable $s$ rather than $z$ in this context. If $s$ is real, the sum is convergent for $s > 1$ and uniformly convergent on $s \geq a$ for any $a > 1$, as one can see from the integral test, for example. If $s$ is complex, then $|n^{-s}| = n^{-\mathrm{Re}(s)}$, so the sum is absolutely convergent for $\mathrm{Re}(s) > 1$, uniformly on $\mathrm{Re}(s) \geq a$ for any $a > 1$. In particular, it can be differentiated term by term, showing that $\zeta(s)$ is analytic on $\mathrm{Re}(s) > 1$.

The zeta function is the cornerstone of applications of analytic methods to number theory. These applications derive from the infinite product representation

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \qquad (\mathrm{Re}(s) > 1). \tag{26}$$

To understand why (26) holds, consider the finite product over primes less than $N$, and expand each factor in a geometric series, valid because $|p^{-s}| < 1$ for $\mathrm{Re}(s) > 1$:

$$\prod_{\substack{p \text{ prime} \\ p<N}} \frac{1}{1 - p^{-s}} = \prod_{\substack{p \text{ prime} \\ p<N}} (1 + p^{-s} + p^{-2s} + \cdots).$$

Multiplying this out gives the sum of $n^{-s} = p_1^{-k_1 s} \cdots p_m^{-k_m s}$ over all positive integers $n$ which have a factorization $n = p_1^{k_1} \cdots p_m^{k_m}$ using only primes $p_i < N$. Letting $N$ go to infinity enlarges the sum to $\zeta(s) = \sum_n n^{-s}$.

Among the things that can be proven using (26) is the Prime Number Theorem, which asserts that the number of primes $p < N$ is asymptotic to $N/\ln(N)$. More precise versions of this estimate would follow from the *Riemann hypothesis*, perhaps the most famous unsolved conjecture in all of mathematics. We will discuss the Riemann hypothesis briefly at the end of these notes.

The proof of the Prime Number Theorem is too involved to go into here. Instead we will limit ourselves to more modest goals, namely (i) to constuct an analytic continuation of $\zeta(s)$ to the entire complex plane except for $s = 1$, where it has a simple pole, and (ii) to derive the famous *functional equation*

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-(1-s)/2}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{27}$$

which this analytic continuation satisfies. Our method will be the same as in the classic textbook *Complex Analysis*, by Ahlfors.

Our first observation is that the change of variables $u = nx$ yields the identity

$$\int_0^\infty x^{s-1}e^{-nx}\,dx = n^{-s}\int_0^\infty u^{s-1}e^{-u}\,du = n^{-s}\Gamma(s)$$

for $\mathrm{Re}(s) > 0$. Summing over all $n$ and using the geometric series

$$\sum_{n=1}^\infty e^{-nx} = \frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^x-1},$$

we get

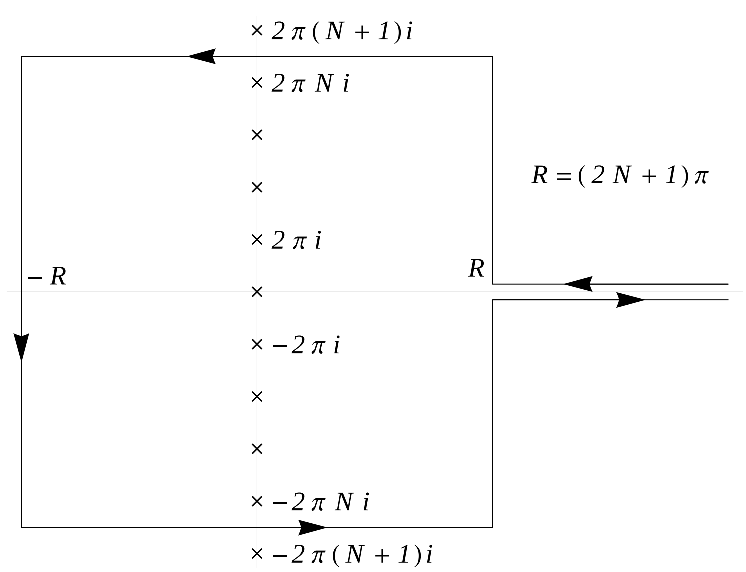
$$\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx = \zeta(s)\Gamma(s) \qquad (\mathrm{Re}(s) > 1).$$

One can justify taking the summation inside this improper integral by truncating the geometric series to finitely many terms and verifying that the integral of the remainder term goes to zero as $n \to \infty$, for $\mathrm{Re}(s) > 1$.

To understand the last integral better, we relate it to a family of contour integrals

$$\int_{C_N} \frac{(-z)^{s-1}}{e^z-1}\,dz \tag{28}$$

over contours $C_N$ for each integer $N \geq 0$, as shown here.



In (28) we use the principal branch $(-z)^{s-1} = e^{(s-1)\,\mathrm{Log}(-z)}$. As in (13), this branch has values $(-z)^{s-1} = e^{-i\pi(s-1)}x^{s-1}$ and $(-z)^{s-1} = e^{i\pi(s-1)}x^{s-1}$, respectively, along the upper and lower sides of the branch cut on the positive real axis.

For $N > 0$ we take $R = (2N+1)\pi$, so that the top and bottom sides of the square portion of the contour pass half-way between the poles at $\pm 2\pi N i$ and $\pm 2\pi(N+1)i$.

For $N = 0$, the situation is a little different. The contour $C_0$ encloses only the singularity at the origin (which is a branch point, not a pole), and we can evaluate the integral by letting $R$ go to zero. The integral on the square portion of $C_0$, which is bounded by a constant times $R \cdot R^{\mathrm{Re}(s)-1} = R^{\mathrm{Re}(s)}$, goes to zero as $R \to 0$ if $\mathrm{Re}(s) > 0$. This leaves the integrals along the branch cut, giving

$$\int_{C_0} \frac{(-z)^{s-1}}{e^z - 1}\,dz = (e^{i\pi(s-1)} - e^{-i\pi(s-1)}) \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx = 2i\sin(\pi(s-1))\zeta(s)\Gamma(s)$$

for $\mathrm{Re}(s) > 1$. Using (14), we can also write this as

$$\zeta(s) = \frac{i}{2\pi}\Gamma(1-s)\int_{C_0} \frac{(-z)^{s-1}}{e^z - 1}\,dz. \tag{29}$$

Now, instead of letting $R$ approach zero, we can keep it fixed, taking $R = \pi$, for instance. On any interval $a \le \mathrm{Re}(s) \le b$, the contour integral in (29) converges uniformly at the infinite ends. This justifies differentiating (29) inside the integral sign, showing that the right hand side defines an analytic function for all $s$ except positive integers, where $\Gamma(1-s)$ has poles. However, we already know that $\zeta(s)$ is analytic for real $\mathrm{Re}(s) > 1$, so (29) defines an analytic continuation of $\zeta(s)$ to all $s \ne 1$.

From here on, $\zeta(s)$ will denote this analytic continuation.

As one application of (29), observe that if $s$ is an integer, then $(-z)^{s-1}$ is actually analytic across what would otherwise be the branch cut on the positive real axis. In this case we can evaluate the integral in (29) using the residue theorem, obtaining

$$\zeta(-n) = (-1)^n\, n!\ \mathrm{Res}_{z=0}\left(\frac{z^{-n-1}}{e^z - 1}\right) \tag{30}$$

for integers $n \ge 0$. The function $1/(e^z - 1)$ has a simple pole at $z = 0$ and therefore a Laurent series expansion of the form

$$\frac{1}{e^z - 1} = z^{-1}\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

The coefficients $B_n$ are called *Bernoulli numbers*. Here are the first several values:

$$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_3 = 0,\ B_4 = -1/30,\ B_5 = 0,\ B_6 = 1/42.$$

Formula (30) can be expressed in terms of the Bernoulli numbers as

$$\zeta(-n) = (-1)^n B_{n+1}.$$

One can verify with a little algebra that $1/2 + 1/(e^z - 1)$ is an odd function, which shows that $B_n = 0$ for $n$ odd and greater than 1. Hence $\zeta(s) = 0$ for $s = -2, -4, -6, \ldots$. These zeroes are called the *trivial zeroes* of the zeta function.

For $s = 1$, the residue theorem gives the value $2\pi i B_0 = 2\pi i$ for the contour integral in (29). Since $\Gamma(1-s)$ has a simple pole with residue $-1$ at $s = 1$, it follows that $\zeta(s)$ has a simple pole with residue 1 at $s = 1$.

Our remaining goal is to derive the functional equation (27). For this we observe that, by the residue theorem, the difference between the integrals (28) over contours $C_N$ and and $C_0$ arises from the poles at $\pm 2\pi n i$ for $n = 1, \ldots, N$. More precisely, we have

$$\int_{C_N} \frac{(-z)^{s-1}}{e^z - 1}\, dz \; - \int_{C_0} \frac{(-z)^{s-1}}{e^z - 1}\, dz$$
$$= 2\pi i \sum_{n=1}^{N} \left( \mathrm{Res}_{z=2\pi n i} \left( \frac{(-z)^{s-1}}{e^z - 1} \right) + \mathrm{Res}_{z=-2\pi n i} \left( \frac{(-z)^{s-1}}{e^z - 1} \right) \right).$$

The function $1/(e^z - 1)$ has residue 1 at $z = 0$ and therefore, since its periodic, also at $z = e^{\pm 2\pi n i}$ for all $n$. This implies

$$\mathrm{Res}_{z=2\pi n i} \left( \frac{(-z)^{s-1}}{e^z - 1} \right) = e^{i\pi(1-s)/2} (2\pi n)^{s-1}, \quad \mathrm{Res}_{z=-2\pi n i} \left( \frac{(-z)^{s-1}}{e^z - 1} \right) = e^{i\pi(s-1)/2} (2\pi n)^{s-1},$$

giving

$$\int_{C_N} \frac{(-z)^{s-1}}{e^z - 1}\, dz - \int_{C_0} \frac{(-z)^{s-1}}{e^z - 1}\, dz = i\,(2\pi)^s\, 2\cos(\frac{\pi}{2}(s-1)) \sum_{n=1}^{N} n^{s-1}. \tag{31}$$

We now examine what happens to the integrals along each segment of the contour $C_N$ as $N$ and $R = (2N+1)\pi$ go to infinity, assuming that $\mathrm{Re}(s) = \sigma < 0$. Along the top edge of the square, $e^z = e^{x+(2N+1)\pi i} = -e^x$, and $|z| \geq R$, giving

$$\left| \frac{(-z)^{s-1}}{e^z - 1} \right| = \frac{|z|^{\sigma-1}}{e^x + 1} < |z|^{\sigma-1} \leq R^{\sigma-1},$$

since $\sigma - 1 < 0$. The integral on this edge is therefore bounded by $2R \cdot R^{\sigma-1} = 2R^\sigma$, which goes to zero as $R \to \infty$. The same reasoning applies to the integral on the bottom edge of the square.

Along the left side of the square, $e^z = e^{-R} e^{iy}$ has $|e^z| = e^{-R}$ small compared to 1, so $|z^{s-1}/(e^z - 1)|$ is again bounded by a constant times $R^{\sigma-1}$. Along the right side, $|e^z| = e^R$ is large compared to 1, and $|z^{s-1}/(e^z - 1)|$ is bounded by a constant times $e^{-R} R^{\sigma-1}$. Hence the integrals on these edges also go to zero.

Finally, the integrals along the branch cut have the form

$$e^{\pm i\pi(s-1)} \int_R^\infty \frac{x^{s-1}}{e^x - 1}\, dx,$$

so they too go to zero as $R \to \infty$. Our conclusion is that

$$\lim_{N\to\infty} \int_{C_N} \frac{(-z)^{s-1}}{e^z - 1}\, dz = 0$$

if $\mathrm{Re}(s) < 0$. Taking the limit as $N \to \infty$ in (31), and using (29) for the value of the integral on $C_0$, we obtain

$$\frac{2\pi i \zeta(s)}{\Gamma(1-s)} = i\,(2\pi)^s\, 2\cos(\frac{\pi}{2}(s-1))\zeta(1-s),$$

or

$$\zeta(s) = (2\pi)^{s-1}\, 2\cos(\frac{\pi}{2}(s-1))\Gamma(1-s)\zeta(1-s). \tag{32}$$

Although we derived this identity assuming that $\mathrm{Re}(s) < 0$, it follows that it holds for all $s \neq 1$, since both sides are analytic. Note that on the right hand side, the poles of $\Gamma(1-s)$ at positive even integers $s$ and the pole of $\zeta(1-s)$ at $s = 0$ are cancelled by zeroes of $\cos(\frac{\pi}{2}(s-1))$, while the poles of $\Gamma(1-s)$ at odd integers $s > 1$ are cancelled by the the trivial zeroes of $\zeta(1-s)$. Alternatively, we can regard (32) as providing another way to see that $\zeta(s) = 0$ for $s = -2, -4, -6, \ldots$.

Our last step is to use Legendre's duplication formula (24) to obtain

$$\Gamma(1-s) = 2^{-s}\pi^{-1/2}\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2})$$

and (14) to obtain

$$\cos(\frac{\pi}{2}(s-1)) = \sin(\pi\frac{s}{2}) = \frac{\pi}{\Gamma(s/2)\Gamma(1-s/2)}.$$

Substituting these into (32), we get

$$\zeta(s) = \pi^{s-1/2}\,\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1-s),$$

which is equivalent to the functional equation (27).

We conclude with a brief discussion of the Riemann hypothesis. The product representation (26) implies that $\zeta(s)$ has no zeros in the half plane $\mathrm{Re}(s) > 1$, and the functional equation (27) then implies that its only zeroes in the half plane $\mathrm{Re}(s) < 0$ are the trivial zeroes at $s = -2, -4, -6, \ldots$. Thus all non-trivial zeroes of $\zeta(s)$ lie in the *critical strip* $0 \leq \mathrm{Re}(s) \leq 1$. The functional equation further implies that the locations of the zeroes in the critical strip are symmetric about its midline $\mathrm{Re}(s) = 1/2$. The Riemann hypothesis asserts, conjecturally, that all zeroes of $\zeta(s)$ in the critical strip are actually on the line $\mathrm{Re}(s) = 1/2$.

It turns out (for reasons involving more advanced techniques of complex analysis) that the product representation (26) implies that the distribution of prime numbers is strongly influenced by the location of the nontrivial zeroes of $\zeta(s)$. To be specific, the Prime Number Theorem can be formulated a little more precisely as

$$N - \sum_{\substack{p \text{ prime} \\ p<N}} \ln(p) = o(N), \tag{33}$$

where the notation $f(N) = o(g(N))$ means $\lim_{N\to\infty} f(N)/g(N) = 0$. In other words, the difference between $N$ and the sum of the logarithms of all primes less than $N$ is small compared to $N$, for large $N$. It can be shown that if all zeroes of $\zeta(s)$ lie in the region $\mathrm{Re}(s) \leq \alpha$ (where $1/2 \leq \alpha \leq 1$), then we can replace (33) with the stronger estimate

$$N - \sum_{\substack{p \text{ prime} \\ p<N}} \ln(p) = o(N^{\alpha+\epsilon}),$$

for any $\epsilon > 0$. The Riemann hypothesis then implies that this estimate holds with $\alpha = 1/2$, that is, the difference between $N$ and the sum of the logarithms of all primes less than $N$ is not only small compared to $N$, but grows essentially no faster than $\sqrt{N}$.