Math 185—Introduction to Complex Analysis
Haiman, Summer 2014

Final Exam

You may consult the class textbook and your own notes. No other printed materials or eletronic devices may be used.

1. (a) [6 pts] Find the Laurent series around $z_0 = 0$ of the function $z^3 e^{1/z}$.

(b) [6 pts] Find the residue $\mathrm{Res}_{z=0}(z^3 e^{1/z})$.

(c) [5 pts] What type of singularity does $z^3 e^{1/z}$ have at $z_0 = 0$: removable, essential, or pole? If it is a pole, what is its order?

2. (a) [4 pts] Verify that $u(x, y) = x^3 y - xy^3$ is a harmonic function.

(b) [7 pts] Find a harmonic conjugate $v(x, y)$ of $u(x, y)$.

(c) [7 pts] Find an entire function $f(z)$ such that $u(x, y) = \mathrm{Re}(f(z))$. Express $f(z)$ directly in terms of z, not in terms of x and y.

3. [18 pts] Evaluate the integral

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx,$$

where we understand the integrand as extending to a continuous function with value $\lim_{x\to 0} \sin(x)/(x(x^2 + 1)) = 1$ at $x = 0$.

4. (a) [5 pts] Show that the equation $e^z = 3z$ has at least one real solution in the interval $[0, 1]$.

(b) [10 pts] Prove that this real solution is the only solution in the unit disk $|z| \leq 1$.

5. [14 pts] Find the maximum and minimum values of $\left|\frac{z}{z-2}\right|$ on the unit disk $|z| \leq 1$. Briefly justify your answer.

6. Let D be the domain between the two circles $|z - 1/2| = 1/2$ and $|z - 1| = 1$.

(a) [1 pt] Sketch D.

(b) [5 pts] Show that the transformation $w = 1/z$ maps D and its boundary circles (excluding the point $z = 0$) to the strip between two parallel lines in the w plane, and determine those lines.

(c) [6 pts] Solve the following Dirichlet problem: find a harmonic function $T(x, y)$ on D, which extends continuously to $T = 0$ on the smaller boundary circle and $T = 1$ on the larger one, except at $z = 0$.

(d) [6 pts] Suppose the function $T(x, y)$ represents a steady state temperature. Describe (geometrically, in words, not formulas) the isotherms and the heat flow curves, and add some of them to your sketch of D.