1. (a) $\sum_{n=0}^{\infty} \frac{1}{n!} z^{3-n}$

(b) The z^{-1} term in the series is at $n=4$, with coefficient

$\mathrm{Res}_{z=0} (z^3 e^{1/z}) = \frac{1}{4!} = \frac{1}{24}$

(c) Essential singularity.

2. (a) $u_x = 3x^2y - y^3 \quad u_{xx} = 6xy$

$u_y = x^3 - 3xy^2 \quad u_{yy} = -6xy$

$u_{xx} + u_{yy} = 0.$

(b) To get $v_y = u_x$, integrate u_x with respect to y, obtaining

$v(x,y) = \frac{3}{2}x^2y^2 - \frac{1}{4}y^4 + F(x).$

Then equate v_x with $-u_y$:

$$3xy^2 + F'(x) = -x^3 + 3xy^2$$

$$F'(x) = -x^3$$

$$F(x) = -\frac{1}{4}x^4. \quad (+C, \text{ but we can choose } C=0)$$

$v(x,y) = -\frac{1}{4}x^4 + \frac{3}{2}x^2y^2 - \frac{1}{4}y^4$

Incidentally, by solving for v satisfying Cauchy-Riemann equations with u, in this way we also show again that u is harmonic, i.e. this solution to (b) includes (a).

(c) $f(x+iy) = u(x,y) + iv(x,y)$ can be written as

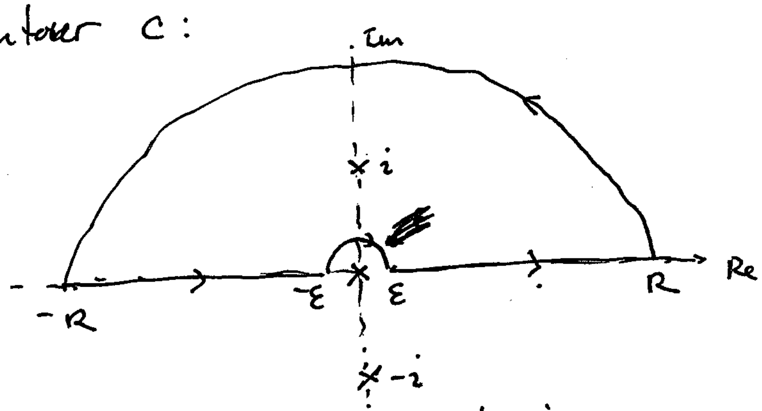
$$-\frac{i}{4}\left(x^4 + 4ix^3y + 6i^2x^2y^2 + 4i^3xy^2 + i^4y^4 \right)$$

$$= -\frac{i}{4} z^4 .$$

3. The integrand is an even function, so

$$\int_0^{\infty} \frac{\sin x}{x(x^2+1)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,dx = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}\,dx .$$

We apply the residue theorem to $\int_C \frac{e^{iz}}{z(z^2+1)}\,dz$ on the indented contour C:



(× marks poles of $\frac{e^{iz}}{z(z^2+1)}$)

The contour encloses the pole at i, giving

$$\int_C \frac{e^{iz}}{z(z^2+1)}\,dz = 2\pi i \,\mathrm{Res}_{z=i} \frac{e^{iz}}{z(z^2+1)} .$$

Factoring $\frac{e^{iz}}{z(z^2+1)} = \frac{e^{iz}}{z(z+i)(z-i)}$ we find its residue at $z=i$ to be $\frac{e^{-1}}{i(2i)} = \frac{-e^{-1}}{2}$, so

$$\mathrm{Im}\int_C \frac{e^{iz}}{z(z^2+1)}\,dz = -\pi/e .$$

Now we evaluate the contribution to $\mathrm{Im}\int_C \frac{e^{iz}}{z(z^2+1)}\,dz$ from each segment of the contour, as $R\to\infty$ and $\varepsilon\to 0$. On the large arc, since e^{iz} is bounded in the upper half-plane, the integral is bounded by a constant times R/R^3, so $\to 0$ as $R\to\infty$ (you could invoke Jordan's Lemma for this, but it is stronger than needed). On the real axis, the contribution to $\mathrm{Im}\int_C \frac{e^{iz}}{z(z^2+1)}\,dz$ is the desired integral $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,dx$.

On the small arc, as $\varepsilon\to 0$, the contribution goes to $-\frac{1}{2}\,2\pi i\,\mathrm{Res}_{z=0}\frac{e^{iz}}{z(z^2+1)}$ (minus sign because the orientation is clockwise).

This residue is $\frac{e^{i0}}{0^2+1} = 1$, so the contribution to $\mathrm{Im}\int_C \frac{e^{iz}}{z(z^2+1)}\,dz$ is $-\pi$.

Hence $-\pi + \int_{-\infty}^{\infty}\frac{\sin x}{x(x^2+1)}\,dx = -\pi/e$, giving $\int_0^{\infty}\frac{\sin x}{x(x^2+1)}\,dx = \frac{1}{2}\left(\pi - \pi/e\right)$

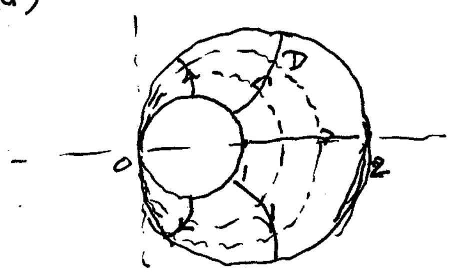
$$= \frac{\pi}{2}\left(1-e^{-1}\right).$$

4. (a) Since $e^{x}-3x$ has value 1 at $x=0$ and $e-3<0$ at $x=1$, and is real and continuous on $[0,1]$, there is an $x_0 \in [0,1]$ such that $e^{x_0}-3x_0=0$, i.e. $e^{x_0}=3x_0$

(b) On $|z|\le 1$ we have $\mathrm{Re}(z)\le 1$, and therefore $|e^z| = e^{\mathrm{Re}(z)} \le e$. We also have $|3z| = 3 > e$. By Rouché's theorem, it follows that $3z-e^z$ has the same number of zeroes in $|z|\le 1$ as $3z$ does, namely one. Thus $z=x_0$ is the only solution of $e^z=3z$ in $|z|\le 1$.

5. The minimum is $\left|\frac{z}{z-2}\right|=0$ at $z=0$. By the maximum modulus principle, the maximum occurs on the boundary circle $|z|=1$. On this circle, $\left|\frac{z}{z-2}\right| = \frac{1}{|z-2|}$, which is maximized when $|z-2|$ is minimized, that is, at the point on $|z|=1$ closest to 2, which is $z=1$. So the maximum is $\left|\frac{z}{z-2}\right| = 1$ at $z=1$.

6. (a) and sketch of (d)



¦ isotherms | flow lines

(b) $w=\frac{1}{z}$ is a Möbius transformation, and it maps 0 to ∞, so it must map the boundary circles to lines. Since 1 and $\frac{1}{2}+i\frac{1}{2}$ are on the small circle, the 1 and $1-i=\left(\frac{1}{2}+i\frac{1}{2}\right)^{-1}$ are on the corresponding line. So the small circle maps to $\mathrm{Re}(w)=1$. Similarly, the large circle maps to $\mathrm{Re}(w)=\frac{1}{2}$, and since $z=\frac{3}{2}$ maps to $w=\frac{2}{3}$, the domain D must map to the region between these two vertical lines.

(c) In the w plane, $T(u,v)=2(1-u)$ is harmonic, $T(\frac{1}{2},v)=1$ on the line $\mathrm{Re}(w)=\frac{1}{2}$, and $T(1,v)=0$ on the line $\mathrm{Re}(w)=1$. Hence $T(x,y)=2\left(1-\mathrm{Re}\left(\frac{1}{z}\right)\right)=2\left(1-\frac{x}{x^2+y^2}\right)$ is the required solution.

(d) In the w plane the isotherms are vertical lines in the strip between $\mathrm{Re}(w)=\frac{1}{2}$ and $\mathrm{Re}(w)=1$, so in the z plane they are circles passing through 0 in between the large and small boundary circles. The flow lines in the w plane are horizontal line segments, so in the z plane they are circular arcs meeting both boundary circles at right angles.