Math 110 Spring 2012
HW 10 Solutions

Exercises

7.6 By the spectral theorem, V has an orthonormal basis consisting of eigenvectors e_i of T, say $Te_i = \lambda_i e_i$. Then $\text{range}(T) = \text{span}(\{e_i : \lambda_i \neq 0\})$. But $T^* e_i = \bar{\lambda}_i e_i$, so $\text{range}(T^*) = \text{span}(\{e_i : \bar{\lambda}_i \neq 0\}) = \text{range}(T)$.

7.7 We can describe $\text{range}(T)$ as in 7.6, above. Then $\text{range}(T^k) = \text{span}(\{e_i : \lambda_i^k \neq 0\}) = \text{range}(T)$. Similarly, $\text{null}(T^k) = \text{span}(\{e_i : \lambda_i^k = 0\}) = \text{null}(T)$.

7.8 The conditions say that $(1,2,3)$ and $(2,5,7)$ are eigenvectors of T with eigenvalues $0, 1$; in particular, with different eigenvalues. If T were self-adjoint, this would imply that the two vectors are orthogonal, but $\langle (1,2,3), (2,5,7) \rangle = 33 \neq 0$. [It is understood that we are using the Euclidean inner product on $\mathbb{R}^3$.]

7.10 T is diagonalizable, by the spectral theorem, and its eigenvalues satisfy $\lambda^9 = \lambda^8$, i.e. $\lambda^8(\lambda - 1) = 0$. Hence all eigenvalues are 0 or 1, so T is a projection and $T^2 = T$. It's self-adjoint (in fact, an orthogonal projection) because the eigenvalues are real.

7.11 Diagonalize T and take for S any operator whose matrix is $\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$ where the matrix of T is $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ and $\alpha_i^2 = \lambda_i$. This is possible, since every complex number has a square root.

7.13 True, by same argument as in 7.11 and using the fact that every real number has a cube root.

7.15 "If" follows from the spectral theorem. For "only if," let $e_1, \ldots, e_n$ be a basis consisting of eigenvectors of T and define $\langle u, v \rangle = \sum a_i \bar{b}_i$ where $u = \sum a_i e_i$, $v = \sum b_i e_i$, so that $\underline{e}$ becomes an $\perp$-normal basis.

7.17 If S, T are positive, then $(S+T)^* = S^* + T^* = S+T$, so $S+T$ is self-adjoint, and $\langle (S+T)v, v\rangle = \langle Sv, v\rangle + \langle Tv, v\rangle \geq 0$ for all v, so $S+T$ is positive.

7.18 $(T^k)^* = (T^*)^k = T^k$, so T^k is self-adjoint. The eigenvalues of T^k are λ_i^k, where λ_i are the eigenvalues of T, so they are ≥ 0, hence T^k is positive.

7.19 If $\langle Tv, v\rangle > 0$ for all v, then $\operatorname{null}(T) = 0$ (since $v \in \operatorname{null}(T)$ implies $\langle Tv, v\rangle = \langle 0, v\rangle = 0$), so T is invertible. Since T is positive, let e_i be a basis of V consisting of eigenvectors of T, say $Te_i = \lambda_i e_i$, where $\lambda_i \geq 0$ (by spectral theorem and definition of positivity). If T is invertible, then all $\lambda_i > 0$. If $v = \sum a_i e_i$, then $\langle Tv, v\rangle = \sum \lambda_i |a_i|^2$. This is > 0 unless all $a_i = 0$, i.e., unless $v = 0$.

7.20 True. For every orthonormal basis (e_1, e_2) of $\mathbb{F}^2$, the operator $Te_1 = -e_1$, $Te_2 = e_2$ is self-adjoint with $T^2 = I$. This operator determines $\operatorname{span}(e_1) = \operatorname{null}(T+I)$. By Gram-Schmidt, $\operatorname{span}(e_1)$ can be any 1-dimensional subspace of $\mathbb{F}^2$, (i.e., every such subspace contains the first member of some orthonormal basis). Since $\mathbb{F}^2$ has infinitely many 1-dimensional subspaces (note $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ — $\mathbb{F}$ is not a finite field!), there are infinitely many distinct operators of the above form T.

7.23 The matrices of T, $\sqrt{T^*T}$, and S in the standard basis of $\mathbb{F}^3$ are (it doesn't matter whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ since the solution is real):

$$T: \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \quad T^*T: \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \sqrt{T^*T}: \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad S: \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

because T^*T happens to be diagonal

Problems

1. I'll prove this assuming either that T is self-adjoint, or, in the case $\mathbb{F} = \mathbb{C}$, normal. In either case we can find an orthonormal basis $e_1, \dots, e_n$ of V consisting of eigenvectors of T, say $Te_i = \lambda_i e_i$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of T. Let $\alpha = \min |\lambda_i|$. If $\|v\| = 1$, and we express v in terms of the basis $(e_1, \dots, e_n)$ as

$$v = a_1 e_1 + \dots + a_n e_n,$$

then $Tv = a_1 \lambda_1 e_1 + \dots + a_n \lambda_n e_n$

$$\|Tv\|^2 = \sum |a_i|^2 |\lambda_i|^2 \geq \alpha^2 \sum |a_i|^2 = \alpha^2, \text{ since } \sum |a_i|^2 = \|v\| = 1,$$

and $|\lambda_i|^2 \geq \alpha^2$ for all i. This shows

$$\min_{\|v\|=1} \|Tv\| \geq \alpha.$$

To prove equality, let i be an index for which $\alpha = |\lambda_i|$ and take $v = e_i$. Then $\|v\| = 1$, and $\|Tv\| = |\lambda_i| = \alpha$.

2. In the case $\mathbb{F} = \mathbb{R}$, $T - \lambda I$ is self-adjoint. In the case $\mathbb{F} = \mathbb{C}$, $(T - \lambda I)^* = T^* - \bar{\lambda} I$, and since T is self-adjoint, this is equal to $T - \bar{\lambda} I$, which commutes with $T - \lambda I$, so $T - \lambda I$ is normal. By hypothesis, $\min_{\|v\|=1} \|(T - \lambda I)v\| < \varepsilon$, so $T - \lambda I$ has an eigenvalue β with $|\beta| < \varepsilon$ by Problem 1. Then $\lambda' = \lambda + \beta$ is an eigenvalue of T, and $|\lambda' - \lambda| = |\beta| < \varepsilon$.

[If you only proved Problem 1 for self-adjoint operators, you will only be able to use it to solve problem 2 in the case $\mathbb{F} = \mathbb{R}$.]