

Math 110 - Fall 2009 - Haiman
Problem Set 13 Solutions

① a) Suppose $x = tu$, $y = su$. Then $|\langle x, y \rangle| = |t\bar{s}\langle u, u \rangle| = |s||t|\|u\|^2$. On the other side, $\|x\| = |t|\|u\|$ and $\|y\| = |s|\|u\|$, so $\|x\|\|y\| = |s||t|\|u\|^2$.

b) Gram-Schmidt applied to (x, y) produces first $u_1 = x$ and then u_2 orthogonal to u_1 . Replacing these with unit vectors $u = u_1/\|u_1\|$, $v = u_2/\|u_2\|$ gives an orthonormal basis of $\text{Span}(\{x, y\})$ in which $x = tu$ with $t = \|u_1\|$. Let $y = au + bv$. Then

$$\langle x, y \rangle = \langle tu, au + bv \rangle = t\bar{a}$$

$$\|x\| = |t|$$

$$\|y\| = \langle au + bv, au + bv \rangle = \sqrt{|a|^2 + |b|^2}$$

$$\text{since } \langle u, u \rangle = \langle v, v \rangle = 1 \\ \langle u, v \rangle = 0$$

So Cauchy-Schwarz reduces to

$$|t||a| \leq |t| \sqrt{|a|^2 + |b|^2}$$

which is true because all quantities here are real numbers ≥ 0 , and $\sqrt{|a|^2 + |b|^2} \geq \sqrt{|a|^2} = |a|$.

② Use the inner product on continuous functions $[a, b] \rightarrow \mathbb{C}$ given by $\langle f, g \rangle = \int_a^b f(t)\overline{g(t)} dt$, and apply Cauchy-Schwarz with $x = f$, $y = 1$ to get

$$|\langle f, 1 \rangle| \leq \|x\| \|y\| \\ \left| \int_a^b f(t) dt \right| \leq \sqrt{\int_a^b |f(t)|^2 dt} \sqrt{b-a} \quad \leftarrow \text{note } \|1\|^2 = \int_a^b 1 \cdot 1 dt = b-a$$

③ Ex. 28: We define $[\cdot, \cdot] = \text{Re} \langle \cdot, \cdot \rangle$.

Then i) $[x, y]$ is linear in x because $[ax + x_2, y] =$

$$\text{Re} \langle ax + x_2, y \rangle = \text{Re} (a \langle x, y \rangle + \langle x_2, y \rangle) = a \text{Re} \langle x, y \rangle + \text{Re} \langle x_2, y \rangle,$$

for $a \in \mathbb{R}$.

ii) $[x, y] = [y, x]$ because $\langle x, y \rangle = \overline{\langle y, x \rangle} \Rightarrow \text{Re} \langle x, y \rangle = \text{Re} \langle y, x \rangle$

iii) $[x, x] = \langle x, x \rangle$ since $\langle x, x \rangle$ is real, so $[x, x] \geq 0$ and $[x, x] = 0 \Rightarrow x = 0$.

In addition, $[x, ix] = \text{Re} \langle x, ix \rangle = \text{Re} (-i \langle x, x \rangle) = 0$ since $\langle x, x \rangle$ is real.

④ The inner product is $\text{tr} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}^T = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$.

Gram-Schmidt gives the ortho basis of $\text{Span}(S)$:

$$v_1 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \quad \langle v_1, v_1 \rangle = 2^2 + 2^2 + 2^2 + 1^2 = 13$$

$$v_2 = \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix} - \frac{\langle \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \rangle}{13} \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix} - \begin{pmatrix} 6 & 6 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix} \quad \langle v_2, v_2 \rangle = 5^2 + 2^2 + 4^2 + 2^2 = 49$$

$$v_3 = \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix} - \frac{\langle \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix}, v_1 \rangle}{13} v_1 - \frac{\langle \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix}, v_2 \rangle}{49} v_2$$

$$= \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix} + 2 \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} + 0 v_2$$

$$= \begin{pmatrix} 8 & -8 \\ 7 & -14 \end{pmatrix} \quad \langle v_3, v_3 \rangle = 8^2 + 8^2 + 7^2 + 14^2 = 373$$

An orthonormal basis is then $\{ v_1/\sqrt{13}, v_2/7, v_3/\sqrt{373} \}$.

The Fourier coefficients of $A = \begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}$ are then

$$\langle A, v_1/\sqrt{13} \rangle = 65/\sqrt{13} = 5\sqrt{13}$$

$$\langle A, v_2/7 \rangle = -98/7 = -14$$

$$\langle A, v_3/\sqrt{373} \rangle = 373/\sqrt{373} = \sqrt{373}$$

$$\text{These give } A = 5\sqrt{13} \cdot \frac{v_1}{\sqrt{13}} - 14 \cdot \frac{v_2}{7} + \sqrt{373} \frac{v_3}{\sqrt{373}}$$

$$= 5v_1 - 2v_2 + v_3 = 5 \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} - 2 \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix} + \begin{pmatrix} 8 & -8 \\ 7 & -14 \end{pmatrix}$$

which you can check is correct. [Note that the apparently magical cancellation of square roots here is simply a consequence of the textbook authors having chosen for A a matrix which is a linear combination v_1, v_2, v_3 with integer coefficients.]

⑤ The orthogonal projection of $h(t) = e^t$ on $P_2(\mathbb{R})$ in the given inner product will be

$$g(t) = \langle e^t, u_1 \rangle u_1 + \langle e^t, u_2 \rangle u_2 + \langle e^t, u_3 \rangle u_3$$

with u_1, u_2, u_3 as in Example 5, p. 346.

The inner products are given by the integrals

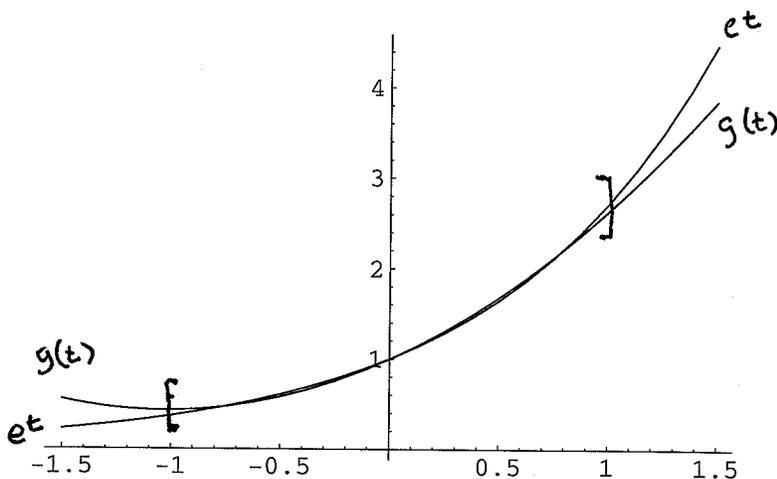
$$\langle e^t, u_1 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^1 e^t dt = \frac{1}{\sqrt{2}} (e - e^{-1})$$

$$\langle e^t, u_2 \rangle = \sqrt{\frac{3}{2}} \int_{-1}^1 t e^t dt = \sqrt{\frac{3}{2}} \cdot 2e^{-1}$$

$$\langle e^t, u_3 \rangle = \sqrt{\frac{5}{8}} \int_{-1}^1 (3t^2 - 1) e^t dt = \sqrt{\frac{5}{8}} (2e - 14e^{-1})$$

$$\begin{aligned} \text{Then } g(t) &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} (e - e^{-1}) + \sqrt{\frac{3}{2}} \sqrt{\frac{3}{2}} (2e^{-1} t) + \sqrt{\frac{5}{8}} \sqrt{\frac{5}{8}} (2e - 14e^{-1}) (3t^2 - 1) \\ &= \frac{1}{2} (e - e^{-1}) + 3e^{-1} t + \frac{5}{8} (2e - 14e^{-1}) (3t^2 - 1) \\ &= \frac{15}{4} (e - 7e^{-1}) t^2 + 3e^{-1} t + \frac{3}{4} (-e + 11e^{-1}) \end{aligned}$$

Here's a plot showing how well $g(t)$ fits the graph of e^t on $[-1, 1]$



⑥ By the definition of W^\perp , every vector $w \in W$ is orthogonal to all vectors ~~in~~ in W^\perp , so $W \subseteq (W^\perp)^\perp$. We need to prove the reverse inclusion $(W^\perp)^\perp \subseteq W$.

So suppose $v \in (W^\perp)^\perp$, i.e. $\langle v, x \rangle = 0$ for every $x \in W^\perp$.

By Theorem 6.6 we have $v = u + z$ with $u \in W$, $z \in W^\perp$.

If $x \in W^\perp$, then $\langle u, x \rangle = 0$, so $\langle v, x \rangle = \langle z, x \rangle$. But $\langle v, x \rangle = 0$ by hypothesis, so $\langle z, x \rangle = 0$. Applying this with $x = z \in W^\perp$, we get $\langle z, z \rangle = 0$, hence $z = 0$, hence $v = u \in W$, which is what we wanted to prove.

⑦ (i) To fit a linear function, we take

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{pmatrix} \quad y = \begin{pmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{pmatrix} \quad \text{and solve for } x = \begin{pmatrix} c \\ d \end{pmatrix}$$

to minimize the error $E = \|Ax - y\|$. This gives the equations $A^*Ax = Ay$, i.e.

$$\begin{pmatrix} 165 & 25 \\ 25 & 5 \end{pmatrix} x = \begin{pmatrix} 220 \\ 34 \end{pmatrix},$$

with solution $x = \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 5/4 \\ 1/20 \end{pmatrix}$. The best fit line is then

$y = \frac{5}{4}t + \frac{1}{20}$. At the data points $t_i = (1, 3, 5, 7, 9)$

this gives $(\frac{9}{5}, \frac{43}{10}, \frac{34}{5}, \frac{93}{10}, \frac{59}{5}) = (1.8, 4.3, 6.7, 9.3, 11.8)$.

The ~~errors~~ individual errors $Ax - y$ are $(-\frac{1}{5}, \frac{2}{10}, -\frac{1}{5}, \frac{2}{10}, -\frac{1}{5})$,

and the norm of this vector gives $E = \sqrt{3/10} \approx .548$

(ii) To fit a quadratic polynomial, take

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \\ 49 & 7 & 1 \\ 81 & 9 & 1 \end{pmatrix}, \quad \text{leading to the equations } A^*Ax = Ay:$$

$$\begin{pmatrix} 9669 & 1225 & 165 \\ 1225 & 165 & 25 \\ 165 & 25 & 5 \end{pmatrix} x = \begin{pmatrix} 1626 \\ 220 \\ 34 \end{pmatrix}, \quad \text{with solution}$$

$$x = \begin{pmatrix} 1/56 \\ 15/14 \\ 239/280 \end{pmatrix}, \quad \text{for a best fit polynomial } y = \frac{t^2}{56} + \frac{15}{14}t + \frac{239}{280}.$$

Evaluated at the data points $(1, 3, 5, 7, 9)$,

$$\text{this gives } \left(\frac{68}{35}, \frac{148}{35}, \frac{233}{35}, \frac{323}{35}, \frac{418}{35} \right)$$

$$\approx (1.94, 4.23, 6.66, 9.23, 11.94).$$

The vector of individual errors $Ax - y$ is $\approx (-.06, .23, -.34, .23, -.06)$,

$$\text{with norm } E = \|Ax - y\| = \sqrt{\frac{8}{35}} \approx .48$$

Note that the greater flexibility allowed by the quadratic fit results in a smaller total error.