

Math 110 - Linear Algebra - Fall 2009
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PS 12 Solutions

① a) $J_n \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is the sum of the columns of J_n , thus equal to $\begin{pmatrix} n \\ \vdots \\ n \end{pmatrix}$. So $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = n$.

b) All the rows of J_n are equal and non-zero, so $\text{rank}(J_n) = \dim(\text{span}(\text{rows})) = 1$. Hence $\text{nullity}(J_n) = n-1$.

c) By (a), J_n has eigenspace E_n with $\dim E_n \geq 1$.

By (b), J_n has eigenspace E_0 with $\dim E_0 = n-1$.

Since $(n-1)+1=n$, it follows that J_n is diagonalizable with a basis consisting of one eigenvector in E_n and $n-1$ in E_0 .

$$\text{So } Q^{-1} J_n Q = D = \begin{pmatrix} n & 0 & \dots & 0 \\ 0 & \ddots & & 0 \end{pmatrix}.$$

d) From (c) it's immediate that $\det(J_n - \lambda I_n) = (-\lambda)^{n-1} \cdot (n-\lambda)$.

e) Plug $\lambda=1$ into $\det(J_n - \lambda I_n)$ to get $\det(Z_n) = (-1)^{n-1}(n-1)$.

In particular, for $n > 1$ this is $\neq 0$, so Z_n is invertible (where the field is understood to be \mathbb{R} or \mathbb{Q} , so we don't have $n=0$ in \mathbb{N} for a positive integer n).

f) $\det(Z_n - \lambda I_n) = \det(J_n - (\lambda+1) I_n) = (-(\lambda+1))^{n-1} (n-1-\lambda)$.

g) Z_n is diagonalizable, with $Q^{-1} Z_n Q = \begin{pmatrix} n-1 & & & \\ -1 & \ddots & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$.

From this it's clear that if $f(t) = (t+1)(t-n+1)$ then $f(Z_n) = 0$.

h) By (g), $Z_n^2 + (2-n)Z_n + (1-n)I_n = 0$. Multiply by Z_n^{-1} to get $(1-n)Z_n^{-1} = -Z_n + (n-2)I_n$

$$Z_n^{-1} = \frac{1}{n-1} Z_n - \frac{n-2}{n-1} I_n = \frac{1}{n-1} \begin{pmatrix} 2-n & 1 & 1 & \cdots \\ 1 & 2-n & 1 & \vdots \\ \vdots & 1 & \ddots & \\ & & & 2-n \end{pmatrix}$$

- (2) By assumption, each W_i has a basis $B_i = \{v_{i,1}, \dots, v_{i,d_i}\}$ consisting of eigenvectors of $T|_{W_i}$, and thus of T . Since $W_1 + \dots + W_k = V$, the union $B_1 \cup \dots \cup B_k$ spans V . Hence V has a basis B which is a subset of $B_1 \cup \dots \cup B_k$, and therefore consists of ~~eigenvectors~~ eigenvectors of T .
- (3) By the definition of T -cyclic subspace, $W = \text{span}(v, T(v), T^2(v), \dots)$. Hence if $w \in W$, then there are scalars a_0, a_1, \dots, a_n such that $w = a_0 v + a_1 T(v) + \dots + a_n T^n(v) = g(T)(v)$, where $g(t) = a_0 + a_1 t + \dots + a_n t^n$. Conversely, if $w = g(T)(v)$, then writing out $g(t) = a_0 + a_1 t + \dots + a_n t^n$, we see that $w = a_0 v + a_1 T(v) + \dots + a_n T^n(v) \in W$. This establishes Exercise 13.

Now suppose $V = W$ and $UT = TU$. By Ex. 13 we can find $g(t)$ such that $g(T)(v) = U(v)$. We now claim that, in fact $g(T)(T^k v) = U(T^k v)$ for all k . To see this, apply T^k to both sides of $g(T)(v) = U(v)$ to get $T^k g(T)(v) = T^k U(v)$, and use the fact that both $g(T)$ and U commute with T , and hence with T^k , to deduce $g(T) T^k(v) = U T^k(v)$.

But by hypothesis, $\{v, T(v), T^2(v), \dots\}$ spans V . We have just shown that the linear transformations $g(T)$ and U agree on each element of this spanning set. Hence they are equal on all of V : $U = g(T)$. This proves Ex. 20.