

Math 110 - Fall 2009 - Hairman
 PS 9 Solutions

(1) Since $Q^{-1}Q = I_n$, $\det(Q^{-1})\det(Q) = 1$, i.e. $\det(Q^{-1}) = \frac{1}{\det(Q)}$.

Then $\det(Q^{-1}AQ) = \det(Q^{-1})\det(A)\det(Q) = \det(A) \frac{\det(Q)}{\det(Q)} = \det(A)$.

Now given $T: V \rightarrow V$, let β and γ be two ordered bases of V and let $Q = [I]_{\gamma}^{\beta}$ be the change of coordinate matrix such that $[T]_{\gamma} = Q^{-1}[T]_{\beta}Q$. Taking $A = [T]_{\beta}$ in the formula above, we get $\det([T]_{\gamma}) = \det([T]_{\beta})$.

(2) (a) Consider the cofactor expansion on the last column.

Every term in the expansion is $\pm x_i^{n-1} \det(V_i)$ where V_i is a Vandermonde matrix in the ~~$n-1$~~ variables with x_i omitted. By induction on n , we can assume $\det(V_i)$ is a homogeneous polynomial of degree $(n-1)(n-2)/2$.

Then every term in $\det(A)$ has degree $n-1 + \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2}$. (The base case for the induction is $n=1$, where $\det(A) = 1$ has degree 0.)

(b) When $x_i = x_j$, the matrix A has two equal rows, so $\det(A) = 0$.

(c) In the cofactor expansion on the ~~last column~~ first row, the only way to get a term with x_i^0 is from the first entry. In turn, in the cofactor $\det(A_{11})$, expanded on its first row, we can only get x_2^1 by taking the first entry. This continues, so that the only term $x_1^0 x_2^1 \dots x_n^{n-1}$ comes from repeatedly expanding each cofactor on the first term in its first row, always with sign $+1$.

So the coefficient of $x_1^0 x_2^1 \dots x_n^{n-1}$ in the result is 1. (This can also be shown by cofactor expansion on last column.)

(d) The product in (b) has $n(n-1)/2$ factors, all of degree 1, so it is homogeneous of degree $n(n-1)/2$, same as $\det(A)$. Thus if $\det(A) = f(x_1, \dots, x_n) \cdot \prod_{i < j} (x_j - x_i)$, then f must be a constant:

$$\det(A) = C \cdot \prod_{i < j} (x_j - x_i).$$

To evaluate C , observe that the monomial $x_1^0 x_2^1 \cdots x_n^{n-1}$ occurs in $\prod_{i < j} (x_j - x_i)$ only once: as the term in which we take the first term from each $(x_j - x_i)$. (We must take first terms from $(x_n - x_1), \dots, (x_n - x_2)$ to get x_n^{n-1} , and the remaining factors form the same product in $n-1$ variables, so we must take their first terms by induction.) Therefore $x_1^0 x_2^1 \cdots x_n^{n-1}$ has coefficient 1 in both $\det(A)$ and $\prod_{i < j} (x_j - x_i)$, hence $C=1$.

(3) $\det(M) = \det(A) \det(C)$. One way to see this is by cofactor expansion on the first column. The terms

~~corresponding to entries below A are zero. For terms $\pm a_{i1} \det(M_{i1})$ where a_{i1} is in A, observe that~~

$$M_{i1} = \begin{pmatrix} A_{i1} & B_i \\ 0 & C \end{pmatrix}, \text{ where } B_i \text{ denotes } B \text{ with the } i^{\text{th}}$$

row deleted. By induction on n , we can assume

$$\det(M_{i1}) = \det(A_{i1}) \det(C), \text{ so } \det(M) = \sum (-1)^{1+i} a_{i1} \det(A_{i1}) \det(C) \\ = \det(A) \det(C).$$

The base case for induction is when the A part is empty, so $M = (C)$ and $\det(M) = \det(C)$.

(4) If $A = \begin{pmatrix} a_{11} & \dots & * \\ 0 & \dots & a_{nn} \end{pmatrix}$ then, as we showed in class,

$\det(A) = a_{11} \dots a_{nn}$. So A invertible $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow$ all $a_{ii} \neq 0$.

(5) Let $A = \begin{pmatrix} A_1 \\ x \\ A_2 \\ y \\ A_3 \end{pmatrix}, B = \begin{pmatrix} A_1 \\ y \\ A_2 \\ x \\ A_3 \end{pmatrix}$

Then $f(B) = f\left(\begin{array}{c} A_1 \\ x+y \\ A_2 \\ x \\ A_3 \end{array}\right) - f\left(\begin{array}{c} A_1 \\ x \\ A_2 \\ x \\ A_3 \end{array}\right)$ by multilinearity,
and the 2nd term is
0 since the matrix has
two equal rows.

Now $f\left(\begin{array}{c} A_1 \\ x+y \\ A_2 \\ x \\ A_3 \end{array}\right) = f\left(\begin{array}{c} A_1 \\ x+y \\ A_2 \\ x+y \\ A_3 \end{array}\right) - f\left(\begin{array}{c} A_1 \\ x+y \\ A_2 \\ y \\ A_3 \end{array}\right)$ by multilinearity,
and 1st term is 0.

Finally, $-f\left(\begin{array}{c} A_1 \\ x+y \\ A_2 \\ y \\ A_3 \end{array}\right) = -f\left(\begin{array}{c} A_1 \\ x \\ A_2 \\ y \\ A_3 \end{array}\right) - f\left(\begin{array}{c} A_1 \\ y \\ A_2 \\ y \\ A_3 \end{array}\right)$. The second term
is 0, and the
first is $-f(A)$.

- (6) a) $L_{P(\pi)}(e_j)$ is the j^{th} column of $P(\pi)$, which is $e_{\pi(j)}$ by definition
 b) $L_{P(\pi)}((x_{\pi(1)}, \dots, x_{\pi(n)})^T) = \sum_{j=1}^n x_{\pi(j)} L_{P(\pi)}(e_j) = \sum_{j=1}^n x_{\pi(j)} e_{\pi(j)} = \sum_{i=1}^n x_i e_i$

$$= (x_1, \dots, x_n)^T$$

- c) Suppose π and π' are the same except for two consecutive values: $\pi(j) = \pi'(j+1)$,

and $\pi(j+1) = \pi'(j)$. Then $P(\pi')$ and $P(\pi)$ differ by switching columns $j \leftrightarrow j+1$, so $\det P(\pi') = -\det P(\pi)$. For $i \neq j$, we have (i, j) an inversion of $\pi \Leftrightarrow (i, j+1)$ is an inversion of π' and $(i, j+1)$ " " " $\pi \Leftrightarrow (i, j)$ " " " π' .

Similarly for $j+1 < k$, (j, k) inv of $\pi \Leftrightarrow (j+1, k)$ inv of π' , $(j+1, k)$ inv of π $\Leftrightarrow (j, k)$ inv of π' . Finally $(j, j+1)$ is inv of $\pi \Leftrightarrow (j, j+1)$ is not an inv of π' . So $\text{inv}(\pi') = \text{inv}(\pi) \pm 1$, and $(-1)^{\text{inv}(\pi')} = -(-1)^{\text{inv}(\pi)}$. Now switch columns repeatedly until $P(\pi') = I_n$, $\pi'(i) = i$, $\text{inv}(\pi') = 0$.