

Math 110 - Linear Algebra - Haiman, Fall '09  
 Problem Set 8 Solutions

- (1) a) True. If  $Ax = b$  has unique solution, then  $N(L_A) = \{0\}$ , and therefore  $\text{rank}(A) = \text{rank}(L_A) = n$ , where  $A$  is  $m \times n$ . But since  $\text{rank}(A) \leq m$  this implies  $m \geq n$ .
- b) False. For example  $\begin{pmatrix} 0 \\ 0 \end{pmatrix} x = \begin{pmatrix} 0 \end{pmatrix}$  is consistent, with  $m=n=1$ . and every  $x \in \mathbb{F}^1$  is a solution.
- c) If  $Ax = b$  is consistent for all  $b$ , then  $L_A$  is onto, i.e.  $\text{rank}(A) = \text{rank}(L_A) = m$ . Since  $\text{rank}(A) \leq n$ , this implies  $m \leq n$ . So this one is true.
- d) False. For example  $\begin{pmatrix} 0 \\ 0 \end{pmatrix} x = \begin{pmatrix} 1 \end{pmatrix}$  is inconsistent, with  $m=n=1$ .

- (2) (a) and (b): To save work, row-reduce the matrix

$$(A \mid b_1 \mid b_2) \quad \text{where (a) is } Ax = b_1 \\ \text{(b) is } Ax = b_2.$$

This gives

$$\left( \begin{array}{ccccc|c|c} 1 & 0 & 0 & -11/34 & 15/34 & 0 & 105/34 \\ 0 & 1 & 0 & -69/34 & 23/34 & 0 & 93/34 \\ 0 & 0 & 1 & 21/17 & 10/17 & 0 & -32/17 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right).$$

From this we see that (a) is inconsistent, and the solution set of (b) is

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11/34 a - 15/34 b + 105/34 \\ 69/34 a - 23/34 b + 93/34 \\ -21/17 a - 10/17 b - 32/17 \\ a \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

③ We have  $A = GR$  where  $R$  is the row-reduced form and  $G$  is some invertible  $3 \times 3$  matrix. Since columns 1, 2 and 4 of  $R$  form an identity matrix  $I_3$ , we must have

$$G = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & -2 \\ 3 & 1 & 0 \end{pmatrix}$$

and therefore  ~~$\text{and}$~~

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & -2 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{pmatrix}$$