

Math 110 - Hainan - Fall 2009
 Problem Set 7 Solutions

① Method: follow the sequence of operations used to transform A into D in the example. For each row operation used, apply the same operation to I_4 . This gives G at the end. For each column operation, apply the same to I_5 , giving F at the end. Using the sequence of operations shown in the example, the result is

$$G = \begin{pmatrix} 0 & 1/4 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 3/4 & -1/2 & 1/4 & 0 \\ 0 & -1/2 & -1/2 & 1 \end{pmatrix} \quad F = \begin{pmatrix} 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Other solutions are possible.

② a) Calculate T on each monomial and take coefficients to get columns of the matrix. (You might also verify that T is linear, although the problem doesn't require you to do so.)

$$T(1) = 2$$

$$T(x) = x + 1 - x = 1$$

$$T(x^2) = x^2 + (1-x)^2 = 2x^2 - 2x + 1$$

$$T(x^3) = x^3 + (1-x)^3 = 3x^2 - 3x + 1$$

$$T(x^4) = x^4 + (1-x)^4 = 2x^4 - 4x^3 + 6x^2 - 4x + 1$$

$$[T]_{\mathcal{B}} = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -3 & -4 \\ 0 & 0 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Using row operations we can change this matrix to

$$\xrightarrow{2} \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now the 1st 3 rows are independent, so $\text{rank}(T) = 3$.

b) Using the table of T values above we can see that the polynomials $2x-1$ and $2x^3-3x^2+x$ belong to $N(T)$. They are independent, since one has an x^3 term and the other doesn't. By the dimension theorem, $\dim(N(T)) = 5 - \text{rank}(T) = 2$, so $\{2x-1, 2x^3-3x^2+x\}$ is a basis. (Other bases of the same space are possible.)

③ Perform row operations on the given matrix and on I_6 :

$$\left(\begin{array}{cccccc|cccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{cccccc|cccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

subtract row 1
from rows 2 through 6

$$\rightarrow \left(\begin{array}{cccccc|cccc} 5 & 0 & 0 & 0 & 0 & 0 & -4 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

add rows 2 through 6
to row 1

$$\rightarrow \left(\begin{array}{cccccc|cccc} 5 & 0 & 0 & 0 & 0 & 0 & -4 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1/5 & 4/5 & -1/5 & -1/5 & -1/5 & -1/5 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1/5 & -1/5 & 4/5 & -1/5 & -1/5 & -1/5 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1/5 & -1/5 & -1/5 & 4/5 & -1/5 & -1/5 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1/5 & -1/5 & -1/5 & -1/5 & 4/5 & -1/5 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 & 4/5 \end{array} \right)$$

subtract $\frac{1}{5} \cdot (\text{row 1})$ from
rows 2 through 6

$$\rightarrow \left(\begin{array}{cccccc|cccc} I_6 & -4/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & -4/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & -4/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & -4/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & -4/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & -4/5 & 1/5 & 1/5 \end{array} \right)$$

multiply row 1 by $\frac{1}{5}$
and the rest by -1

So the inverse matrix is

$$\frac{1}{5} \left(\begin{array}{cccccc} -4 & 1 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & 1 & -4 \end{array} \right)$$

d) Clearly $0 \in S^\perp$.

If $\lambda, \mu \in S^\perp$, then $(\lambda + \mu)(v) = \lambda(v) + \mu(v) = 0$
and $(c\lambda)(v) = c \cdot \lambda(v) = 0$
for all $v \in S$.

So $\lambda + \mu, c\lambda \in S^\perp$. This shows S^\perp is a subspace.

If $S = \text{Span}(v_1, \dots, v_k)$, then $\lambda = a_1\lambda_1 + \dots + a_k\lambda_k$ belongs to S^\perp if and only if $\lambda(v_1) = \dots = \lambda(v_k) = 0$, iff $a_1 = a_2 = \dots = a_k = 0$,
iff $\lambda \in \text{Span}(\lambda_{k+1}, \dots, \lambda_n)$.

e) If V is finite dimensional, say $\dim(V) = n$, and S is a subspace, we can always choose a basis $\{v_1, \dots, v_n\}$ of V s.t.
 $S = \text{Span}(v_1, \dots, v_k)$, where $\dim(S) = k$. Then part (d) shows that $\dim(S^\perp) = n - k = n - \dim(S) = \dim(V) - \dim(S)$.

f) ~~If $\lambda \in R(T)^\perp$~~

We have $\lambda \in R(T)^\perp \Leftrightarrow \lambda(T(v)) = 0 \quad \forall v \in V \Leftrightarrow \lambda T = 0 \text{ in } V^*$
 $\Leftrightarrow \lambda \in N(T^*)$.

g) [Here we are still assuming $T: V \rightarrow W$ where V and W are finite dimensional, with dual $T^*: W^* \rightarrow V^*$.]

By dimension theorem,

$$\begin{aligned} \text{rank}(T^*) &= \dim(W^*) - \dim(N(T^*)) \\ &= \dim(W) - \dim(R(T)^\perp) \quad \text{by (f)} \\ &= \dim(R(T)) \quad \text{by (e)} \\ &= \text{rank}(T) \end{aligned}$$

h) By definition, $\text{rank}(A) = \text{rank}(L_A)$.

By (g), $\text{rank}(L_A) = \text{rank}(L_A^*)$.

By (c), $[L_A^*]_{\beta'}^{\gamma'} = [L_A]^T = A^T$, where β', γ' are dual to the standard bases in $\mathbb{F}^n, \mathbb{F}^m$ (here taking A to be $m \times n$).

By Thm. 3.3, $\text{rank}(L_A^*) = \text{rank}([L_A^*]_{\beta'}^{\gamma'}) = \text{rank}(A^T)$.