

- (1) a) • Contains 0 : if  $T=0$  then  $N(T)=V \supseteq V'$
- Closed under + : if  $V' \subseteq N(T_1)$  and  $V' \subseteq N(T_2)$  we must show  $V' \subseteq N(T_1+T_2)$ . For this, let  $v \in V'$ . Then  $T_1(v)=0$ ,  $T_2(v)=0$  by hypothesis, so  $(T_1+T_2)(v)=0+0=0$ . Hence  $v \in N(T_1+T_2)$ .
  - Closed under scalar mult : if  $V' \subseteq N(T)$  we must show  $V' \subseteq N(aT)$ . For this, let  $v \in V'$ . Then  $T(v)=0$ , so  $(aT)(v)=aT(v)=a \cdot 0=0$ . Hence  $v \in N(aT)$ .
- b) • Contains 0: if  $T=0$  then  $R(T)=\{0\} \subseteq W'$ , since  $W'$  is a subspace.
- Closed under + : Suppose  $R(T_1), R(T_2) \subseteq W'$ . If  $w \in R(T_1+T_2)$ , we have  $w=(T_1+T_2)(v)=T_1(v)+T_2(v)$  for some  $v \in V$ . Then  $T_1(v), T_2(v) \in W'$  by hypothesis, hence  $w \in W'$  since  $W'$  is a subspace. This shows  $R(T_1+T_2) \subseteq W'$ .
  - Closed under scalar  $\cdot$  : Suppose  $R(T) \subseteq W'$ . If  $w \in R(aT)$ , let  $w=aT(v)$ . Then  $w \in W'$ , since  $T(v) \in W'$  and  $W'$  is a subspace. This shows  $R(aT) \subseteq W'$ .
- Remark As the proof in part (a) shows, it is not even necessary to assume that the subset  $V' \subseteq V$  is a subspace. But we do need  $W'$  to be a subspace for (b).
- (2)  $T^2=0$  means  $T(T(v))=0$  for all  $v \in V$ , which is the same as saying  $R(T) \subseteq N(T)$ . Hence  $\text{rank}(T) \leq \text{nullity}(T)$ . Now it follows that
- $$\text{rank}(T) = \frac{\text{rank}(T) + \text{nullity}(T)}{2} \leq \frac{\text{rank}(T) + \text{nullity}(T)}{2} \leq \frac{\text{nullity}(T) + \text{nullity}(T)}{2} = \text{nullity}(T).$$
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 $\dim(V)/2$

③ The simplest solution is to define  $U$  and  $T$  on the standard basis  $\{e_1, e_2\}$  by  $T(e_1) = 0$ ,  $T(e_2) = e_1$ ,  $U(e_1) = 0$ ,  $U(e_2) = e_2$ . Then  $R(T) = \text{Span}(\{e_1\}) \subseteq N(U)$ , so  $UT = 0$ . However,  $TU(e_2) = e_1$ , so  $TU \neq 0$ .

Now take their matrices in the standard basis to get

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

You can check directly that  $AB = 0$  and  $BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = B$ .

[Note that  $TU(e_1) = 0$ , which together with  $TU(e_2) = e_1$ , shows  $TU = T$ , which explains why we got  $BA = B$ .]

$$\textcircled{4} \quad (AB)_{ii} = \sum_{j=1}^m A_{ij} B_{ji}, \text{ so } \text{tr}(AB) = \sum_{i=1}^m \sum_{j=1}^m A_{ij} B_{ji}.$$

Similarly,  $(BA)_{jj} = \sum_{i=1}^m B_{ji} A_{ij}$ , so  $\text{tr}(BA) = \sum_{j=1}^m \sum_{i=1}^m B_{ji} A_{ij} = \text{tr}(AB)$ .

⑤ First, suppose  $\text{rank}(L_A) = 1$ , i.e.  $R(L_A)$  has dimension 1, and is thus spanned by a single vector  $Y \in \mathbb{F}^m$ . (Here (column)

taking  $A$  to be  $m \times n$ ). This means  $Av$  is a scalar multiple of  $Y$  for all  $v \in \mathbb{F}^n$ . In particular, for each  $j$ , the  $j$ th column of  $A$ , which is  $Ae_j$ , is a scalar multiple  $x_j Y$ . This is equivalent to  $A = YX$  with  $X = (x_1, \dots, x_n)$ .

Conversely, suppose  $A = \begin{matrix} Y & X \end{matrix}$ .

Then  $L_A = L_Y L_X$ , so  $R(L_A) = L_Y(R(L_X))$ . Since  $X \neq 0$ ,  $L_X : \mathbb{F}^n \rightarrow \mathbb{F}^1$  has  $R(L_X) = \mathbb{F}^1$ . Then  $R(L_A) = L_Y(\mathbb{F}^1) = \{Y \cdot (a) : a \in \mathbb{F}\} = \{aY\} = \text{Span}(\{Y\})$ . Since  $Y \neq 0$ ,  $\dim(R(L_A)) = 1$ .