

Math 110 - Fall '05 - Haiman
Problem Set 4 Solutions

(1) a) D and $\frac{d}{dx} : P(\mathbb{R}) \rightarrow P(\mathbb{R})$ are both linear, and agree on the basis $\{1, x, x^2, \dots, x^n, \dots\}$. Hence they are the same.

b) If n is even, $D(x^n) = 0$. If n is odd, $D(x^n) = x^{n+1}$, and $n-1$ is even, so $D(x^{n-1}) = 0$. Hence $D(D(x^n)) = 0$ for all n , and since $D(D(f(x)))$ is a linear function of $f(x)$, and $\{1, x, x^2, \dots\}$ is a basis of $P(\mathbb{F}_2)$, it follows that

$$\boxed{D(D(f(x))) = 0} \text{ for all } f(x) \in P(\mathbb{F}_2)$$

c) The operator $E : P(F) \rightarrow P(F)$ defined by $E(f(x)) = g(x, x)$, where $g(x, y) = (f(x) - f(y))/(x-y)$, is easily seen to be linear. We have $E(x^n) = (x^n - y^n)/(x-y) \underset{y \rightarrow x}{=} x^{n-1} y^0 + x^{n-2} y^1 + \dots + x^0 y^{n-1} \underset{y \rightarrow x}{=} n x^{n-1} = D(x^n)$. Since D and E are both linear and agree on the basis, $D(f(x)) = E(f(x))$ for all $f(x) \in P(F)$.

d) Using (c), $D(f(x)g(x)) = h(x, x)$ where $h(x, y) = (f(x)g(x) - f(y)g(y))/(x-y)$.

$$\begin{aligned} f(x)g(x) - f(y)g(y) &= f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y) \\ &= f(x)(g(x) - g(y)) + (f(x) - f(y))g(y), \end{aligned}$$

$$\text{so } h(x, y) = f(x)[(g(x) - g(y))/(x-y)] + [(f(x) - f(y))/(x-y)]g(y).$$

$$\text{Then } h(x, x) = f(x)D(g(x)) + D(f(x))g(x).$$

(2) (e) $\begin{bmatrix} 1 & 0 & & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 1 & 0 & & 0 \end{bmatrix}_{(n \times n)}$ (f) $\begin{bmatrix} 0 & \dots & 1 \\ 1 & \dots & 0 \end{bmatrix}_{(n \times n)}$ (g) $\begin{bmatrix} 1 & 0 & \dots & 0 & 1 \end{bmatrix}_{(1 \times n)}$

(3) W invariant for T means each of $T(v_1), \dots, T(v_k)$ is in the span of $\{v_1, \dots, v_k\}$.

Equivalently, each of $[T(v_1)]_\beta, \dots, [T(v_k)]_\beta$ is a column vector which has zeroes in the last $n-k$ rows.

These are the first k columns of $[T]_\beta$, so the above is equivalent to $[T]_\beta$ having 0's in the lower-left $(n-k) \times k$ rectangle, i.e. the block form indicated in the problem.

The first k entries of $[T(v_i)]_\beta$ (for $i=1, \dots, k$) are its coefficients of the vectors v_1, \dots, v_k , i.e. they form the coordinate vector $[T|_W(v_i)]_{\{v_1, \dots, v_k\}}$. The matrix whose columns are these column vectors is the upper-left block A , that is, $\{T|_W\}_{\{v_1, \dots, v_k\}} = A$.