

### Problem Set 3 Solutions

(1) Use induction on  $k$ . For  $k=0$ , the empty set is independent by definition. For  $k \geq 0$ , we can assume  $S = \{v_1, \dots, v_{k-1}\}$  independent by induction. Since  $v_k \notin \text{Span}(S)$  (and, in particular,  $v_k \notin S$ ), it follows that  $S \cup \{v_k\} = \{v_1, \dots, v_k\}$  is independent, by Thm 1.7 in your book.

(2) a) By problem (1), we can choose an independent sequence  $(v_1, \dots, v_n)$  by choosing each vector in succession, subject to the condition  $v_j \notin \text{Span}(\{v_1, \dots, v_{j-1}\})$  for each  $j$  (when  $j=1$  this means  $v_1 \neq 0$ , since  $\text{Span}(\emptyset) = \{0\}$ ). Then each  $\{v_1, \dots, v_{j-1}\}$  is independent, hence  $\dim(\text{Span}(\{v_1, \dots, v_{j-1}\})) = j-1$ . By PS 2, problem 5, it follows that  $|\text{Span}(\{v_1, \dots, v_{j-1}\})| = 2^{j-1}$ , and  $|V| = 2^n$ , so there are  $2^n - 2^{j-1}$  choices for  $v_j$ . Hence

$$Q(n, k) = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{k-1})$$

b) Part (a) implies that the number of sequences  $(v_1, \dots, v_k)$  that are bases of any given  $k$ -dimensional subspace  $W \subseteq V(\mathbb{F}_2^n)$  is  $Q(k, k)$ . Every independent sequence  $(v_1, \dots, v_k)$  is a basis of one such  $W$ , namely  $W = \text{Span}(\{v_1, \dots, v_k\})$ , and since  $Q(k, k)$  of these are bases for each  $W$ , the number of  $k$ -dimensional subspaces is  $Q(n, k)/Q(k, k)$ .

$$\begin{aligned} c) & (2^{10}-1)(2^{10}-2)(2^{10}-2^2)(2^{10}-2^3)(2^{10}-2^4)/(2^{5-1})(2^{5-2})(2^{5-2^2})(2^{5-2^3})(2^{5-2^4}) \\ & = 109,221,651 \end{aligned}$$

(3) a) Lagrange interpolation implies that there exists  $f(x) \in P_{n-1}(\mathbb{F})$  such that  $(f(c_1), \dots, f(c_n))$  is any specified vector in  $\mathbb{F}^n$ . Since  $P_m(\mathbb{F}) \supseteq P_{n-1}(\mathbb{F})$  for  $m \geq n-1$ , this shows that  $E$  is onto.

b) Since  $E: P_n(\mathbb{F}) \rightarrow \mathbb{F}^n$  is onto,  $\dim(P_n(\mathbb{F})) = n+1$ , and  $\dim(\mathbb{F}^n) = n$ , Thm 2.3 in the book gives  $\text{Nullity}(E) = 1$ . Now  $E(f) = \vec{0}$  means every  $c_i$  is a root of  $f$ , so the space  $N(E)$  of such polynomials has dimension 1. The polynomial  $f(x) = (x-c_1)(x-c_2) \cdots (x-c_n)$  belongs to  $N(E)$  and is not the 0 polynomial, so  $f(x)$  spans  $N(E)$ , i.e. every polynomial  $g(x) \in N(E)$  is a scalar multiple of  $f$ .

(4) Following the hint, define  $T: P_{n-d}(\mathbb{F}) \rightarrow P_n(\mathbb{F})$  by  
 $T(f(x)) = p(x)f(x)$ . Since  $\deg p(x) = d$ , multiplying any  $f(x) \in P_{n-d}(\mathbb{F})$   
by  $p(x)$  gives  $p(x)f(x) \in P_n(\mathbb{F})$ , so the definition makes sense.  
 $T$  is linear because  $p(x)(af(x) + bg(x)) = a p(x)f(x) + b p(x)g(x)$ .  
Finally, if  $f(x) \in P(\mathbb{F})$  has degree  $m > n-d$  then  $p(x)f(x)$   
has degree  $m+d > n$ , so  $p(x)f(x) \notin P_n(\mathbb{F})$ . By definition  
any  $g(x) \in P_n(\mathbb{F})$  divisible by  $p(x)$  is  $g(x) = p(x)f(x)$  for some  
 $f(x)$ , and the preceding sentence shows that  $f(x) \in P_{n-d}(\mathbb{F})$ .  
Hence  $W = R(T)$ , which proves (a) by Thm. 2.1 in the book.

For (b), since  $p(x) \neq 0$  (as its degree is  $=d$ ), we have  
 $f(x) \neq 0 \Rightarrow p(x)f(x) \neq 0$ , or equivalently  $p(x)f(x) = 0 \Rightarrow f(x) = 0$ .

In other words,  $N(T) = \{0\}$ . Then Thm 2.3 gives

$$\begin{aligned} N(T) + R(T) &= \dim(P_{n-d}(\mathbb{F})) = n-d+1, \\ \stackrel{\text{"}}{0} + \stackrel{\text{"}}{R(W)} &= \dim(W) \end{aligned}$$

So  $\dim(W) = n-d+1$ .

(5) a) Since  $\overline{w+z} = \overline{w} + \overline{z}$ , we get  
 $T((w_1, \dots, w_n) + (z_1, \dots, z_n)) = (\bar{w}_1, \dots, \bar{w}_n) + (\bar{z}_1, \dots, \bar{z}_n) = T(\bar{w}) + T(\bar{z})$   
i.e.  $T$  is additive.

b) If  $a \in \mathbb{R}$  is real, then  $\overline{az} = \bar{a}\bar{z} = a\bar{z}$ , so

$$T(a(z_1, \dots, z_n)) = a(\bar{z}_1, \dots, \bar{z}_n) = aT((z_1, \dots, z_n)).$$

Combined with a), this means  $T$  is a linear transformation  
of vector spaces over  $\mathbb{R}$ .

(6) a) Since the rows of  $A+B$  are the sums of a row of  $A$  and corresponding  
row of  $B$ , and since the rows of  $cA$  are  $c$  times the rows of  $A$ ,  
it's clear that  $S(A+B) = S(A) + S(B)$  and  $S(cA) = cS(A)$ .

b) (Assuming  $m > 0$ ) we can get any vector  $\vec{v}$  in  $\mathbb{F}^n$  as the ~~row space~~ sum  
of rows of a matrix with first row  $\vec{v}$  and all other rows  $\vec{0}$ . So  
 $S$  is onto,  $R(S) = \mathbb{F}^n$

c) By definition,  $N(S)$  is the set  $N$  in part (d).

d) Since  $N = N(S)$ ,  $N$  is a subspace by Thm. 2.1, and Thm 2.3 gives

$$\dim(N) + \dim(\mathbb{F}^n) = \dim(\underset{n}{M}_{m \times n}), \text{ hence } \dim(N) = (m-1)n.$$

⑦ a) By Thm 2.3,

$$\begin{aligned}\dim R(T) &= \dim(v) - \dim(N(T)) \\ &\leq \dim(v) && \text{since } \dim(N(T)) \geq 0 \\ &< \dim(w) && \text{by assumption.}\end{aligned}$$

Hence  $R(T) \neq W$ , i.e.  $T$  is not onto.

b) Similarly,

$$\begin{aligned}\dim N(T) &= \dim(v) - \dim(R(T)) \\ &\geq \dim(v) - \dim(w) && \text{since } R(T) \subseteq W, \\ &&& \text{so } \dim(R(T)) \leq \dim W \\ &&& \text{by Thm. 1.11} \\ &> 0 && \text{by assumption.}\end{aligned}$$

Hence  $N(T) \neq \{0\}$ , so  $T$  is not 1-to-1 by Thm 2.4

(or, more directly, because  $T^{-1}(\{0\}) = N(T)$  by definition and  $N(T) \neq \{0\}$  implies  $N(T)$  has more than one element.)