

# Math 110 Fall '09

## Problem Set 1 Solutions

① a)  $+$ :  $(a+bi) + (a'+b'i) = (a+a') + (b+b')i$  ;  
with  $a+a', b+b' \in \mathbb{Q}$

$-$ :  $-(a+bi) = -a + (-b)i$ ;  $-a, -b \in \mathbb{Q}$ .

$\times$ :  $(a+bi)(a'+b'i) = (aa'-bb') + (ab'+a'b)i$  ;  
with  $aa'-bb', ab'+a'b \in \mathbb{Q}$

b) The field identities hold in  $\mathbb{Q}[i]$  because they hold in  $\mathbb{C}$ . We just have to check that if  $a+bi \neq 0$  is in  $\mathbb{Q}[i]$ , then so is its inverse. But

$$(a+bi)^{-1} = \frac{a-bi}{a^2+b^2}, \text{ and we have } \frac{a}{a^2+b^2}, \frac{b}{a^2+b^2} \in \mathbb{Q}.$$

② Axioms (VS1-2) hold for  $M_{m \times n}(\mathbb{F})$  because  $+$  is defined entrywise. The 0 element (VS3) is the  $m \times n$  matrix with all entries 0; the additive inverse (VS4) of  $A \in M_{m \times n}(\mathbb{F})$  is  $-A$ , computed entrywise. (VS5) is clear from the definition of  $cA$ . (VS6) follows from the definition of  $cA$  and the fact that  $\cdot$  is associative in  $\mathbb{F}$ . (VS7-8) follow from the distributive law in  $\mathbb{F}$ .

③ (VS 1-4) follow from the same axioms for  $V$ .

For (VS5) note that  $1 \in \mathbb{C}$  is  $1+0i$ :

[Graded: 10 points]

$$(1+0i)(v_1, v_2) \stackrel{\text{def}}{=} (1v_1 + 0v_2, 1v_2 + 0v_1) = (v_1, v_2)$$

For (VS 6), calculate

$$\begin{aligned} & (a+bi)(c+di)(v_1, v_2) \\ &= ((ac-bd) + (ad+bc)i)(v_1, v_2) \\ &= ((ac-bd)v_1 + (ad+bc)v_2, (ac-bd)v_2 + (ad+bc)v_1). \end{aligned}$$

Compare

$$\begin{aligned} & (a+bi)((c+di)(v_1, v_2)) \\ &= (a+bi)(cv_1 - dv_2, cv_2 + dv_1) \\ &= (a(cv_1 - dv_2) - b(cv_2 + dv_1), a(cv_2 + dv_1) + b(cv_1 - dv_2)) \\ &= ((ac-bd)v_1 - (bc+ad)v_2, (ac-bd)v_2 + (ad+bc)v_1) \end{aligned}$$

which agrees with the previous calculation.

(VS 7-8) are verified by similar (somewhat easier) calculation and comparison.

④ a)  $W_1 + W_2$  is a subspace because (using Theorem 1.3)

•  $0 = 0 + 0 \in W_1 + W_2$

• If  $x = x_1 + x_2, y = y_1 + y_2 \in W_1 + W_2$ , then

$$x + y = \underbrace{(x_1 + y_1)}_{W_1} + \underbrace{(x_2 + y_2)}_{W_2} \in W_1 + W_2$$

• If  $x = x_1 + x_2 \in W_1 + W_2$  then  $cx = \underbrace{cx_1}_{W_1} + \underbrace{cx_2}_{W_2} \in W_1 + W_2$

~~Thus~~ We have  $W_1 \subseteq W_1 + W_2$  because  $x \in W_1 \Rightarrow x = \underbrace{x}_{W_1} + \underbrace{0}_{W_2} \in W_1 + W_2$

and similarly,  $W_2 \subseteq W_1 + W_2$

b) Suppose  $W \subseteq V$  is a subspace and  $W_1, W_2 \subseteq W$ .

For any  $x = x_1 + x_2 \in W_1 + W_2$ , we have  $x_1 \in W_1 \subseteq W$ ,

$x_2 \in W_2 \subseteq W$ , and hence  $x = x_1 + x_2 \in W$  since  $W$  is

closed under  $+$ .

⑤  
[Graded: 5 pts  
per part]

a)  $W+W=W$ . We have  $W \subseteq W+W$  by ④(a) and  $W+W \subseteq W$  by ④(b) (with  $W_1=W_2=W$ ).

b) Not true. For example take  $V = \mathbb{R}^2$ ,  
 $W_1 = \{(x, 0) : x \in \mathbb{R}\}$ ,  $W_2 = \{(0, x) : x \in \mathbb{R}\}$ ,  
 $W_3 = \{(x, x) : x \in \mathbb{R}\}$ . These are the subspaces  
spanned by the single vectors  $w_1 = (1, 0)$ ,  $w_2 = (0, 1)$ ,  
 $w_3 = (1, 1)$ , respectively.

Now  $W_1 \cap W_3 = \{0\} = W_2 \cap W_3$ , so

$$(W_1 \cap W_3) + (W_2 \cap W_3) = \{0\}.$$

But  $W_1 + W_2 = V$ , since every  $(x, y) \in V$  can be  
expressed as  $(x, 0) + (0, y)$ . So

$$(W_1 + W_2) \cap W_3 = W_3 \neq \{0\}.$$

c) Suppose  $w = w_1 + w_2 = w'_1 + w'_2$  are two such expressions.

Then  $w_1 - w'_1 + w_2 - w'_2 = w - w = 0$ , so

$$w_1 - w'_1 = w'_2 - w_2.$$

But  $w_1 - w'_1 \in W_1$ ,  $w'_2 - w_2 \in W_2$ , so both sides of  
the above equation are in  $W_1 \cap W_2 = \{0\}$ .

Thus  $w_1 - w'_1 = 0 = w'_2 - w_2$ , i.e.,  $w'_1 = w_1$  and  $w'_2 = w_2$ .

So the expression is unique.

⑥ a) Use Theorem 1.3

[Graded:  
5 pts per  
part]

•  $f(x) = 0$  satisfies  $f(-x) = 0$ , so  $0 \in W$

• If  $f(x) = f(-x)$ ,  $g(x) = g(-x)$ , then  $(f+g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f+g)(-x)$ , so  $f+g \in W$

• If  $f(x) = f(-x)$  then  $cf(x) = cf(-x)$ , so  $cf \in W$ .

b) Let  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ .

Then  $f(-x) = (-1)^n a_n x^n + (-1)^{n-1} a_{n-1} x^{n-1} + \dots + a_0$ ,

i.e. odd-degree terms change sign, while even degree terms are the same as in  $f(x)$ . Hence

$f(x) = f(-x)$  iff  $a_k = 0$  for all odd  $k$ , that is,

iff  $f(x)$  is a linear combination of the monomials  $\{x^n \mid n \text{ even}\}$ .

⑦ a) Solve the system of equations

$$2a + 1b = 5$$

$$0a + 3b = 3$$

$$-1a + 3b = 1$$

$$1a + 2b = 4$$

to discover that  $\vec{w} = 2\vec{u} + 1\vec{v}$ .

b) and c) give the same system of equations as in (a), and therefore also we have  $\vec{w} = 2\vec{u} + 1\vec{v}$  in each case.