

1. In the vector space $M_{2 \times 2}(\mathbb{R})$, let S be the subspace consisting of symmetric matrices, and let U be the subspace consisting of matrices A such that

$$(1 \ 1) A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (0).$$

Find a basis of $S \cap U$.

If A is symmetric, say $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, then the equation $(1 \ 1) A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (0)$ means $a+2b+c=0$. Therefore $A \in S \cap U$ iff (a, b, c) is in the 2-dimensional space of solutions of $a+2b+c=0$, with any two linearly independent solutions providing a basis, e.g.

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

2. Show that if there exists an invertible matrix $A \in M_{n \times n}(\mathbb{R})$ such that $A^T = -A$, then n must be even (A^T denotes the transpose of A).

Changing the sign of one row changes the sign of $\det(A)$, so changing the sign of all n rows, we get $\det(-A) = (-1)^n \det(A)$. Now $\det(A^T) = \det(A)$, so if $A^T = -A$, then $\det(A) = (-1)^n \det(A)$. If A is invertible, $\det(A) \neq 0$, so we can divide by $\det(A)$ to get $(-1)^n = 1$, i.e., n is even.

3. Let $T: V \rightarrow V$ be a linear transformation. Prove that $W = N(T^2)$ is an invariant subspace for T , and that the restriction T_W of T to W satisfies $(T_W)^2 = 0$.

If $v \in N(T^2)$, then $T^2(T(v)) = T^3(v) = T(T^2(v)) = T(0) = 0$.

This shows $T(v) \in N(T^2)$, so $N(T^2)$ is invariant.

By definition of T_W , if $v \in N(T^2)$, then $(T_W)^2(v) = T_W(T_W(v)) = T(T(v)) = T^2(v) = 0$. So $(T_W)^2 = 0$.

4. Let $W_{n,k}$ denote the subspace of $P_n(\mathbb{R})$ consisting of polynomials $f(x)$ such that $f(1), f(2), \dots, f(k)$ are all equal to zero. Find $\dim(W_{n,k})$, in terms of n and k . You will probably need to consider the cases $k \leq n$ and $k \geq n+1$ separately.

If $k \geq n+1$, then $f(x) \in W_{n,k}$ implies f is a polynomial of degree $\leq n$ with $>n$ roots, hence $f=0$. Therefore $W_{n,k} = 0$, and so $\dim(W_{n,k}) = 0$, in this case.

If $k \leq n$, the linear map $E: P_n(\mathbb{R}) \rightarrow \mathbb{R}^k$ defined by $E(f) = (f(1), f(2), \dots, f(k))$ is surjective, by Lagrange interpolation. So $\text{rank}(E) = k$. Now $W_{n,k} = N(E)$, hence $\dim(W_{n,k}) = \text{nullity}(E) = \dim(P_n(\mathbb{R})) - \text{rank}(E) = n+1-k$ in this case.

5. (a) Find the real 3×3 matrix M such that M has eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

with corresponding eigenvalues $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1$.

Put $Q = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and calculate $Q^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.

$$\text{Then } M = Q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} Q^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Since you can (and should) check your answer by calculating Mv_1, Mv_2 , and Mv_3 , we will not be very generous with partial credit for arithmetic errors when grading this problem.

6. For which real numbers t does the system of linear equations

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

have a non-zero solution x ? Justify your answer.

We need the matrix A of coefficients to be singular, i.e., $\det(A)=0$. Expanding by cofactors on last row or column gives $\det(A) = t \cdot 1 - 3 \cdot 2 + 1 \cdot 1 = t - 5$, so $t=5$.

Another way to solve the problem is to require $\text{rank}(A) < 3$. Since the 1st 2 rows are independent, this means we need the third row to be a linear combination $a(1 \ 1 \ 1) + b(1 \ 2 \ 3)$.

Solving for a and b to get 1 and 3 in the first two entries, we find $a=-1, b=2$, so the third row must be $(1 \ 3 \ 5)$, i.e. $t=5$.

7. Either diagonalize the matrix

$$B = \begin{pmatrix} -2 & 0 & 1 \\ 4 & 2 & -3 \\ -4 & 0 & 2 \end{pmatrix}$$

or show that B is not diagonalizable.

The characteristic polynomial is

$$\det \begin{pmatrix} -2-t & 0 & 1 \\ 4 & 2-t & -3 \\ -4 & 0 & 2-t \end{pmatrix} = t^2(2-t).$$

Hence the eigenvalues are $\lambda=2$, with multiplicity 1, and $\lambda=0$, with multiplicity 2. For diagonalizability, we need the $\lambda=0$ eigenspace, i.e. $N(B)$, to have dimension 2. But the 1st two rows of B are independent, so $\text{rank}(B) \geq 2$, hence $\text{rank}(B)=2$ and $\dim N(B)=1$. This shows that B is not diagonalizable.

8. Use the Cayley-Hamilton Theorem to find real numbers a, b, c such that $A^{-1} = aI + bA + cA^2$, where

$$A = \begin{pmatrix} 1 & 2 & -5 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{pmatrix}.$$

The characteristic polynomial is $p(t) = (1-t)(3-t)(-2-t)$

$= -t^3 + 2t^2 + 5t - 6$. By Cayley-Hamilton,

$$-A^3 + 2A^2 + 5A - 6I = 0,$$

hence $-\frac{1}{6}A^3 + \frac{1}{3}A^2 + \frac{5}{6}A = I$

$$A\left(-\frac{1}{6}A^2 + \frac{1}{3}A + \frac{5}{6}I\right) = I$$

$$A^{-1} = -\frac{1}{6}A^2 + \frac{1}{3}A + \frac{5}{6}I$$

9. Using the standard inner product on \mathbb{R}^3 in which the standard basis $\{e_1, e_2, e_3\}$ is orthonormal, find the orthogonal projection of the vector $e_1 = (1, 0, 0)$ on the subspace $W = \text{Span}(\{(1, 1, 1), (1, 2, 3)\})$.

First get an orthogonal basis of W using Gram-Schmidt:

$$x_1 = w_1 = (1, 1, 1) \quad \langle x_1, x_1 \rangle = 3$$

$$x_2 = w_2 - \frac{\langle w_2, x_1 \rangle}{\langle x_1, x_1 \rangle} x_1 = (1, 2, 3) - \frac{6}{3} (1, 1, 1) = (-1, 0, 1) \quad \langle x_2, x_2 \rangle = 2$$

Then the projection of e_1 is given by

$$\cancel{\langle e_1, x_1 \rangle}{\cancel{\langle x_1, x_1 \rangle}} x_1 + \cancel{\langle e_1, x_2 \rangle}{\cancel{\langle x_2, x_2 \rangle}} x_2$$

$$\cancel{\langle e_1, x_1 \rangle}{\cancel{\langle x_1, x_1 \rangle}} = \frac{1}{3} (1, 1, 1) + \frac{-1}{2} (-1, 0, 1) = \left(\frac{5}{6}, \frac{1}{3}, -\frac{1}{6} \right).$$

10. Let V be a finite-dimensional inner product space, and let $\{v_1, \dots, v_n\}$ be an orthonormal basis of V . Let $T: V \rightarrow V$ be a linear operator such that $T^*T = I$, where T^* denotes the adjoint of T . Prove that $\{T(v_1), \dots, T(v_n)\}$ is an orthonormal basis of V .

$$\begin{aligned} \text{We have } \langle T(v_i), T(v_j) \rangle &= \langle v_i, T^*T(v_j) \rangle \\ &= \langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This shows that $\{T(v_1), \dots, T(v_n)\}$ is an orthonormal set, hence linearly independent, hence a basis, since $\dim(V) = n$. (Or, since $T^*T = I$, T is invertible and therefore $\{T(v_1), \dots, T(v_n)\}$ is a basis.)