

1. (10 points each) Determine whether each of the following assertions is *true* or *false*. Give a brief explanation for each answer (full proof is not required).

(a) If  $A \in M_{n \times n}(\mathbb{R})$  and  $n$  is odd, then  $A$  has an eigenvector.

TRUE  $p(\lambda) = \det(A - \lambda I)$  is a polynomial of degree  $n$  over  $\mathbb{R}$ . Since  $n$  is odd,  $p(\lambda)$  has at least one real root, so  $A$  has at least one eigenvalue and corresponding eigenvector.

(b) If  $A$  is an  $n \times n$  matrix with two distinct eigenvalues  $\lambda_1$  and  $\lambda_2$ , and  $\dim(E_{\lambda_1}) = n - 1$ , then  $A$  is diagonalizable.

TRUE Since  $\lambda_2$  is an eigenvalue,  $\dim(E_{\lambda_2}) \geq 1$ . Hence  $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) \geq n$ . We always have  $\leq$  here, so  $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) = n$ . By the criterion for diagonalizability, this implies  $A$  is diagonalizable.

(c) If  $A$  and  $B$  are invertible matrices in  $M_{n \times n}(\mathbb{F})$ , then  $\det(ABA^{-1}B^{-1}) = 1$ .

$$\begin{aligned}\text{TRUE } \det(ABA^{-1}B^{-1}) &= \det(A)\det(B)\det(A^{-1})\det(B^{-1}) \\ &= \det(A)\det(B) \frac{1}{\det(A)} \frac{1}{\det(B)} \\ &= 1.\end{aligned}$$

2. (20 points) Give a correct version of the following incorrect statement:

Given a matrix  $A \in M_{m \times n}(\mathbb{F})$  and a column vector  $b \in \mathbb{F}^m$ , there is a vector  $x_0 \in \mathbb{F}^n$  such that the solution set of the system of equations

$$Ax = b$$

consists of the vectors  $x = x_0 + v$  for all  $v$  in the nullspace  $N(A)$ .

If the system is consistent, and  $x_0$  is a solution,  
then the solution set consists of the vectors  $x = x_0 + v$   
for all  $v \in N(A)$ .

3. (20 points) Express the last entry  $x_4$  of the solution of the system

$$\begin{pmatrix} 4 & 2 & 2 & 1 \\ 3 & 3 & 0 & 0 \\ 2 & 4 & 4 & 1 \\ 3 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

in terms of determinants. Leave the determinants in your answer unevaluated. You may assume that the coefficient matrix of this system is invertible.

$$x_4 = \frac{\det \begin{pmatrix} 4 & 2 & 2 & 1 \\ 3 & 3 & 0 & 0 \\ 2 & 4 & 4 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix}}{\det \begin{pmatrix} 4 & 2 & 2 & 1 \\ 3 & 3 & 0 & 0 \\ 2 & 4 & 4 & 1 \\ 3 & 3 & 0 & 1 \end{pmatrix}}$$

by Cramer's Rule.

4. (30 points) Calculate  $A^{2009}$ , where

$$A = \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} -2-\lambda & 3 \\ -1 & 2-\lambda \end{pmatrix} = \lambda^2 - 4 + 3 = \lambda^2 - 1 = (\lambda+1)(\lambda-1).$$

Find eigenvectors:

$$\lambda = 1 : \begin{pmatrix} -3 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = x_2, v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ will do.}$$

$$\lambda = -1 : \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = 3x_2, v_{-1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ will do}$$

$$\text{Then } Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ where } Q = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \text{ so}$$

$$\begin{aligned} A &= Q \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q^{-1}, \quad A^{2009} = Q \begin{pmatrix} 1^{2009} & 0 \\ 0 & (-1)^{2009} \end{pmatrix} Q^{-1} \\ &= Q \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q^{-1} \\ &= A = \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}. \end{aligned}$$

[Note it is not actually necessary to calculate  $Q^{-1}$  to reach the conclusion that  $A^{2009} = A$ .]