

1. (6 points each) Determine whether each of the following assertions is *true* or *false*. Give a brief explanation for each answer (full proof is not required).

(a) If $D: P_3(\mathbb{R}) \rightarrow \mathbb{R}$ is linear and satisfies $D(1) = 0$, $D(x) = 1$, $D(x^2) = 2$, and $D(x^3) = 3$, then $D(f(x)) = f'(1)$ for all $f(x) \in P_3(\mathbb{R})$.

True. The linear transformations D and $T(f(x)) = f'(1)$ agree on the basis $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$, therefore they are equal.

(b) There exist (non-empty) matrices A and B such that $AB = I$ and $BA = 0$.

False. Solution 1: by homework, $\text{tr}(AB) = \text{tr}(BA)$. But $\text{tr}(I_n) = n$ and $\text{tr}(0) = 0$.

Solution 2: $L_A L_B = I$ implies L_A is onto, and therefore $R(L_A) \neq \{0\}$, and L_B is 1-to-1, from which it follows that $L_B(R(L_A)) \neq \{0\}$. But $L_B(R(L_A)) = R(L_B L_A)$, so this implies $L_B L_A \neq 0$.

(c) If $\{v_1, \dots, v_n\}$ is a basis of V , $\{w_1, \dots, w_n\}$ is a basis of W , and $T: V \rightarrow W$ is a linear transformation such that $T(v_i) = w_i$ for all i , then T is an isomorphism.

True. Let $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{w_1, \dots, w_n\}$. The matrix of T is then $[T]_{\beta}^{\gamma} = I_n$ by hypothesis. Since I_n is an invertible matrix, T is invertible. Alternatively: T is onto since the elements $T(v_i)$ span W , and an onto linear transformation between spaces with $\dim(V) = \dim(W) = n$ is invertible.

(d) For every vector space V , the set of linear transformations $T: V \rightarrow V$ such that $T^2 = 0$ is a subspace of $\mathcal{L}(V, V)$.

False. For a counterexample, take $V = \mathbb{R}^2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $L_A^2 = 0$, $L_B^2 = 0$, but $(L_A + L_B)^2 \neq 0$ since $A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = I_2$.

(e) There exists a sequence of row and column operations that transforms

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

False. The first matrix has rank 3 and the second has rank 2.
But row and column operations preserve rank.

2. (25 points) Suppose V is a finite-dimensional vector space, $T: V \rightarrow V$ is a linear transformation, and β and γ are ordered bases of V . Let P be the change of coordinate matrix such that $P[x]_\beta = [x]_\gamma$ for all $x \in V$. Express each of the matrices $[T]_\gamma$, $[T]_\beta^\gamma$ and $[T]_\gamma^\beta$ in terms of $[T]_\beta$, P and P^{-1} .

We have $P = [I]_\beta^\gamma$, $P^{-1} = [I]_\gamma^\beta$.

$$\text{Therefore } [T]_\gamma = [T]_\gamma^\gamma = [I]_\beta^\gamma [T]_\beta^\gamma [I]_\gamma^\beta = P[T]_\beta P^{-1}$$

$$[T]_\beta^\gamma = [I]_\beta^\gamma [T]_\beta^\beta = P[T]_\beta$$

$$[T]_\gamma^\beta = [T]_\beta^\beta [I]_\gamma^\beta = [T]_\beta P^{-1}$$

3. (20 points) Find A^9 , where

$$A = \begin{pmatrix} \cos(\pi/5) & -\sin(\pi/5) \\ \sin(\pi/5) & \cos(\pi/5) \end{pmatrix}$$

In the standard basis β of \mathbb{R}^2 , A is the matrix

$[R_{\pi/5}]_\beta$ of the rotation through angle $\pi/5$. Then

$$A^9 = [R_{\pi/5}]_\beta^9 = [R_{9\pi/5}]_\beta = [R_{-\pi/5}]_\beta = \begin{pmatrix} \cos(\pi/5) & \sin(\pi/5) \\ -\sin(\pi/5) & \cos(\pi/5) \end{pmatrix}$$

4. (25 points) Invert the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ -3 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & 2 & 0 \end{pmatrix}$$

You need not show every step, but you should indicate enough so we can see your method.

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -3 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \text{switch rows}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -3 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \text{add multiples of row 1 to rows 2 and 4}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & -3 & 1 \end{array} \right) \rightarrow \text{add multiples of rows 2 and 3 to row 4}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cancel{0} & \cancel{1} & -2 & 1 \end{array} \right) \rightarrow \text{multiply row 2 by } -1 \text{ and add row 4 to it}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cancel{0} & \cancel{1} & -2 & 1 \end{array} \right)$$

The inverse matrix is therefore

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since you can, and should, check your answer by multiplying this by the original matrix, we will not be very generous about arithmetic errors when grading this problem.