Math 110 - Fall 2003 - Haiman
Midterm 1 Solutions

1. (5 points each) Determine whether each of the following assertions is *true* or *false*. Give a brief explanation for each answer (full proof is not required).

(a) If a linear transformation $T\colon V \to W$ between finite-dimensional vector spaces is 1-to-1, then $\dim(V) \leq \dim(W)$.

True. Since $\text{nullity}(T) = 0$, $\dim(V) = \text{rank}(T)$ by the dimension theorem. And $\text{rank}(T) \leq \dim(W)$ since $R(T) \subseteq W$.

(b) If $V$ and $W$ are finite-dimensional vector spaces such that $\dim(V) \leq \dim(W)$, and $T\colon V \to W$ is a linear transformation, then $T$ is 1-to-1.

False. A counterexample is the zero map $T: \mathbb{R}^n \to \mathbb{R}^n$ for any $n > 0$.

(c) The set of vectors $(x_1, x_2, x_3, x_4)$ which satisfy $x_1 = x_4$ and $x_2 = x_3$ is a subspace of $\mathbb{R}^4$.

True. The simplest reason is that part (d) is also true.

(d) The set of vectors in part (c) is the nullspace of a linear transformation from $\mathbb{R}^4$ to some vector space over $\mathbb{R}$.

True. It's the nullspace of $T: \mathbb{R}^4 \to \mathbb{R}^2$ defined by $T((x_1, x_2, x_3, x_4)) = (x_1 - x_4, x_2 - x_3)$. (It's easy to check that $T$ is linear, but you need not do so to get full credit on the problem.)

(e) The set of vectors in part (c) is the nullspace of a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}$ (in other words, a linear functional).

False. Since $T$ in part (d) is onto, it has $\text{rank}(T) = 2$, and therefore its nullspace, which is the set of vectors in (c), has dimension 2. But the nullspace of any linear $S: \mathbb{R}^4 \to \mathbb{R}$ has dimension $\geq 3$ by dimension theorem.

(f) $\mathbb{Q}^n$ is a subspace of the vector space $\mathbb{R}^n$ over $\mathbb{R}$. ($\mathbb{Q}$ denotes the field of rational numbers.)

False. Not closed under scalar multiplication by irrational scalars

(g) $\mathbb{Q}^n$ is a subspace of $\mathbb{R}^n$ considered as a vector space over $\mathbb{Q}$ (with the usual addition, and multiplication by rational scalars).

True. Clearly closed under addition, and closed under scalar multiplication since $(ax_1, \ldots, ax_n) \in \mathbb{Q}^n$ if $a$ and all $x_i$ are rational.

2. Let $S$ be the following subset of $P(\mathbb{R})$:

$$S = \{f(x) = x^5 + x^2,\ g(x) = x^5 + 2,\ h(x) = x^3,\ j(x) = x^2 - 2\}$$

(a) (30 points) Find a subset of $S$ which is a basis of Span($S$) and prove that your answer is correct.

There are three possible correct answers: any subset consisting of $h(x)$ and two elements from $\{f(x), g(x), j(x)\}$. I'll prove that $B = \{f(x), g(x), h(x)\}$ is a basis. The proof for the other bases is similar.

First we'll show $\mathrm{Span}(B) = \mathrm{Span}(S)$. Since $B \subseteq S$, $\mathrm{Span}(B) \subseteq \mathrm{Span}(S)$, and to prove $\mathrm{Span}(S) \subseteq \mathrm{Span}(B)$, since $\mathrm{Span}(B)$ is a subspace, it's enough to prove $S \subseteq \mathrm{Span}(B)$. Thus we only need to show that $j(x)$ is in $\mathrm{Span}(B)$, which is true because

$$j(x) = f(x) - g(x).$$

Now we'll show $B$ is linearly independent. Suppose $a, b, c$ are scalars such that

$$af(x) + bg(x) + ch(x) = 0$$

(identically as polynomials). The left hand side is

$$(a+b)x^5 + cx^3 + ax^2 + 2b.$$

For this to be the $0$ polynomial we must have $a = b = c = 0$.

(b) (5 points) Find dim(Span($S$)).

$$\dim(\mathrm{Span}(S)) = |B| = 3.$$

3. (30 points) Let $T\colon V \to W$ be a linear transformation. Prove that if $T$ is 1-to-1, and $v_1, \ldots, v_k \in V$ are linearly independent, then $T(v_1), \ldots, T(v_k)$ are linearly independent.

Suppose $a_1 T(v_1) + a_2 T(v_2) + \ldots + a_k T(v_k) = 0$.

Since $T$ is linear, the left-hand side is equal to

$$T(a_1 v_1 + a_2 v_2 + \ldots + a_k v_k).$$

Since $T$ is 1-to-1, the fact that this is zero implies

$$a_1 v_1 + a_2 v_2 + \ldots + a_k v_k = 0.$$

Finally, since $v_1, \ldots, v_k$ are linearly independent, this implies that all coefficients $a_i$ are zero.

Hence $T(v_1), \ldots, T(v_k)$ are linearly independent.