AFFINE HECKE ALGEBRAS AND POSITIVITY OF LLT AND MACDONALD POLYNOMIALS

I. GROJNOWSKI AND M. HAIMAN

Abstract. We introduce a "hybrid" basis $\{CT_w\}$ of the Hecke algebra of an arbitrary symmetrizable Kac-Moody algebra, depending on the choice of a standard parabolic subgroup W_J of the Weyl group W. We prove that the operator of right multiplication by any Kazhdan-Lusztig basis element has positive matrix coefficients with respect to the hybrid basis. When $W_J = W$ or $W_J = \{1\}$, this reduces, respectively, to a theorem of Springer and Lusztig for the Kazhdan-Lusztig basis, or of Dyer and Lehrer for the standard basis. Our theorem also implies an improvement of the positivity theorem of Kashiwara and Tanisaki for Deodhar's parabolic Kazhdan-Lusztig polynomials, by removing the restriction that W_J be finite.

In the affine A_n case, we obtain as a consequence of our theorem the full positivity theorem for the q-symmetric functions introduced by Lascoux, Leclerc and Thibon, known as LLT polynomials. Special cases of LLT positivity had been proven earlier by Leclerc and Thibon, using the positivity theorem for Deodhar's polynomials. Our result, combined with the combinatorial formula of Haglund, Haiman and Loehr for Macdonald polynomials, yields a new proof of the positivity theorem for Macdonald polynomials. Our methods also provide a definition and positivity theorem for LLT polynomials of other types.

We also give a formula for the expansion of LLT polynomials in terms of generalized Hall-Littlewood polynomials. In certain cases our formula equates a single generalized Hall-Littlewood polynomial with an LLT polynomial. In particular, we prove a conjecture of Shimozono and Weyman equating generalized Hall-Littlewood polynomials indexed by rectangular Young diagrams with LLT polynomials for GL_n.

1. Introduction

Let us briefly recall two well-known positivity theorems in Kazhdan-Lusztig theory. One is the theorem of Springer and Lusztig [19], [25] that the operator of multiplication by a Kazhdan-Lusztig basis element C_v has positive matrix coefficients with respect to the Kazhdan-Lusztig basis $\{C_w\}$ of the Hecke algebra associated to any symmetrizable Kac-Moody Lie algebra. These matrix coefficients are polynomials in $q^{\pm 1}$, so by *positive* we mean coefficient-wise. The second theorem we wish to recall is that of Dyer and Lehrer [4], which asserts (in the finite case, but it is true in general) that the matrix coefficients of the operator C_v with respect to the standard basis $\{T_w\}$ of the Hecke algebra are also positive. In particular, the positivity of Kazhdan-Lusztig polynomials is a special case.

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In this paper we prove a new positivity theorem—Theorem 3.2—which generalizes both of the two aforementioned theorems. Our theorem asserts that the operator of right multiplication by C_v has positive matrix coefficients with respect to a *hybrid basis* $\{CT_w\}$, defined in §3.1, which depends on the choice of a standard parabolic subgroup W_J of the Weyl group W, and interpolates between the standard basis (when $W_J = 1$) and the Kazhdan-Lusztig basis (when $W_J = W$).

We give two applications of our positivity theorem. The first is a new proof and strengthening of a theorem of Kashiwara and Tanisaki [11] on the positivity of Deodhar's parabolic Kazhdan-Lusztig polynomials.

The second and more important application is a proof of the positivity conjecture for *LLT polynomials*, which were defined combinatorially by Lascoux, Leclerc and Thibon in [16]. Combined with the results of Haglund, Loehr and the second author in [6], this yields a second proof of the *Macdonald positivity conjecture*, different from the one based on Hilbert schemes in [8]. These results make use of our positivity theorem for the affine Hecke algebra associated to GL_n. By considering other affine Hecke algebras as well, we discover a natural definition of LLT polynomials associated with any reductive Lie group G, whose coefficients are always positive by our main theorem.

The proof of our positivity theorem uses geometric methods which have been standard in Kazhdan-Lusztig theory for many years. Nevertheless, it seems that both the theorem and the applications are new.

After briefly reviewing some of the required geometric tools in §2, we define the hybrid basis and state and prove our main theorem in §3. The application to parabolic Kazhdan-Lusztig polynomials is in §4. In §5 we define LLT polynomials associated to any reductive Lie group G, then show in §6 that when $G = GL_n$, these essentially coincide with the combinatorial ones defined by Lascoux, Leclerc and Thibon, obtaining the positivity theorems for combinatorial LLT polynomials and Macdonald polynomials as corollaries.

The last section, §7, contains results which are independent of our positivity theorem, but fit naturally into the context of the rest of the paper. Here we give a formula expressing LLT polynomials associated to G in terms of *generalized Hall-Littlewood polynomials* associated to the same group. We also give a criterion which in certain instances implies that a particular generalized Hall-Littlewood polynomial is equal to an LLT polynomial. In the case $G = GL_n$, this enables us to prove a conjecture of Shimozono and Weyman [24] that the generalized Hall-Littlewood polynomial attached to a sequence of rectangular Young diagrams coincides with an LLT polynomial indexed by the same sequence of diagrams.

2. Preliminaries on mixed Hodge modules

2.1. Let $\mathcal{D}_{\mathcal{M}}(X)$ denote the derived category of mixed Hodge modules on a complex algebraic variety X. We will only use the formal properties of the standard functors between these categories, and their interactions with weights, as explained in [23]. A reader who prefers l-adic étale sheaves and Deligne's Weil conjecture machinery will readily transfer things to that context.

A mixed Hodge module A consists of a perverse sheaf A_{rat} with $\mathbb{Q}$ coefficients, a *weight filtration* in A_{rat}, and a filtered structure on the corresponding D-module $\mathrm{RH}(A_{\mathrm{rat}} \otimes_{\mathbb{Q}} \mathbb{C})$, called the *Hodge filtration*, satisfying certain compatibilities. A mixed Hodge module is *pure* if its weight filtration is concentrated in a single weight m. An object A of $\mathcal{D}_{\mathcal{M}}(X)$ is pure of weight m if its i-th cohomology module $H^i(A)$ (corresponding to the i-th perverse cohomology of A_{rat}) is pure of weight $m+i$ for all i.

Let X, Y be varieties, $f\colon X \to Y$ a morphism. There are functors $\boldsymbol{D}\colon \mathcal{D}_{\mathcal{M}}(X) \to \mathcal{D}_{\mathcal{M}}(X)$ (Verdier dual), f_*, $f_!\colon \mathcal{D}_{\mathcal{M}}(X) \to \mathcal{D}_{\mathcal{M}}(Y)$ (direct image; same with proper supports), f^*, $f^!\colon \mathcal{D}_{\mathcal{M}}(Y) \to \mathcal{D}_{\mathcal{M}}(X)$ (inverse image and its dual), and $\boxtimes\colon \mathcal{D}_{\mathcal{M}}(X) \times \mathcal{D}_{\mathcal{M}}(Y) \to \mathcal{D}_{\mathcal{M}}(X \times Y)$ (outer product) which lift their counterparts on the underlying derived categories of constructible sheaves. As usual one defines $\otimes\colon \mathcal{D}_{\mathcal{M}}(X) \times \mathcal{D}_{\mathcal{M}}(X) \to \mathcal{D}_{\mathcal{M}}(X)$ by $A \otimes A' = \delta^*(A \boxtimes A')$, where $\delta\colon X \to X \times X$ is the diagonal morphism. The functors $f^!$ and f_* raise weights, $f_!$ and f^* lower weights, $\boldsymbol{D}$ reverses weights, and $\boxtimes$ adds weights.

2.2. Given a smooth variety X of dimension d and a local system V on X carrying an admissible, polarizable variation of mixed Hodge structure (VMHS), there is a corresponding object V^H in $\mathcal{D}_{\mathcal{M}}(X)$ such that $V^H_{\mathrm{rat}} = V$, with weight and Hodge filtrations induced from those of V. The shift $V^H[d]$ is a mixed Hodge module, pure of weight $m+d$ if V is pure of weight m.

Given any variety X, let $Z \subseteq X$ be an irreducible closed subvariety, $U \subseteq Z$ a smooth open subvariety, and V an irreducible (hence pure) VMHS on U. Then $V^H[\dim(Z)]$ extends uniquely to an irreducible mixed Hodge module $IC_Z(V)$ on Z, whose underlying perverse sheaf $IC_Z(V)_{\mathrm{rat}}$ is the intersection complex of the local system V. Every irreducible mixed Hodge module on X is of the form $(i_Z)_*\, IC_Z(V)$, which we also denote by $IC_Z(V)$. We write $\mathbb{Q}_U$ for the trivial VMHS on U, so $IC_Z(\mathbb{Q}_U)$ is the standard intersection complex on Z.

Define $A(n) = A \boxtimes \mathbb{Q}(n)$, where $\mathbb{Q}(n)$ is the one-dimensional Hodge module on a point with Hodge structure of type $(-n, -n)$. A mixed Hodge module A is *Tate* if all its composition factors are of the form $IC_Z(V)(n)$, where V is an irreducible local system with trivial Hodge structure. An object in $\mathcal{D}_{\mathcal{M}}(X)$ is Tate if its cohomology modules are Tate. The Grothendieck group of Tate mixed Hodge modules on X is a free $\mathbb{Z}[q^{\pm 1}]$-module with basis given by the objects $IC_Z(V)[-\dim(Z)]$, where q acts as the *Tate twist* $A \mapsto A(-1)$. Note that $IC_Z(V)[-\dim(Z)]$, which extends V^H, is pure of weight 0. If A is a pure Tate object of even weight in $\mathcal{D}_{\mathcal{M}}(X)$, it is immediate that the coefficients of its class with respect to the basis objects $IC_Z(V)[-\dim(Z)]$ are polynomials in $q^{\pm 1}$ with *non-negative* coefficients.

2.3. Let $\mathbb{C}^*$ act algebraically on a quasi-projective variety X, and assume that X possesses a $\mathbb{C}^*$-equivariant ample line bundle, or equivalently, that there exists an equivariant immersion $X \hookrightarrow \mathbb{P}(V)$, where $\mathbb{C}^*$ acts linearly on V. Let Z be a connected component of the fixed-point locus $X^{\mathbb{C}^*}$. Recall [1] that the points $x \in X$ such that $\pi(x) = \lim_{t\to 0}\, t \cdot x$ exists and belongs to Z form a subvariety $Y \subseteq X$ called the *attracting variety* to Z, and that the *attracting map* $\pi\colon Y \to Z$ is a morphism.

We will need the following basic result on the purity of hyperbolic localization. For a proof, see [2, Theorem 1, eq. (1), and Remark (4)]. The existence of an equivariant quasi-projective embedding suffices in place of the normality hypothesis in [2], by the paragraph following [2, Lemma 5].

Proposition 2.4. *Let X be a quasi-projective variety with a $\mathbb{C}^*$ action and an equivariant ample line bundle. Let Z be a connected component of $X^{\mathbb{C}^*}$, Y the attracting variety to Z, $\pi\colon Y \to Z$ the attracting map, and $i\colon Y \to X$ the inclusion. Let A be a $\mathbb{C}^*$-equivariant object of $\mathcal{D}_{\mathcal{M}}(X)$. If A is pure of weight n, then so is $\pi_! i^* A$.*

3. Positivity theorem for Hecke algebras

3.1. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra, W its Weyl group, with Coxeter generators $S = \{s_1, \dots, s_n\}$ and Bruhat order $\leq$. The *Hecke algebra* of $\mathfrak{g}$ is the $\mathbb{Z}[q^{\pm 1}]$-algebra $\mathcal{H}$, free as a $\mathbb{Z}[q^{\pm 1}]$-module, with basis $\{T_w : w \in W\}$ satisfying the relations

$$\begin{aligned} T_v T_w &= T_{vw} \quad \text{if } l(vw) = l(v) + l(w) \\ (T_{s_i} - q)(T_{s_i} + 1) &= 0. \end{aligned} \tag{1}$$

There is a $\mathbb{Z}$-linear involution $\overline{\cdot}$ of $\mathcal{H}$ such that $\overline{q} = q^{-1}$ and $\overline{T_w} = T_{w^{-1}}^{-1}$. The *Kazhdan-Lusztig basis*[1] $\{C_w : w \in W\}$ is uniquely determined by the properties [13]

(i) $\overline{C_w} = q^{-l(w)} \overline{C_w}$,
(ii) $C_w = T_w + \sum_{v<w} P_{v,w}(q) T_v$ for polynomials $P_{v,w}(q) \in \mathbb{Z}[q]$ of degree $< \frac{1}{2}(l(w) - l(v))$.

Fix a subset $J \subseteq S$ and the standard parabolic subgroup $W_J \subseteq W$ generated by J. Each right coset $W_J\, x$ has a unique minimum-length representative x, which is also minimal in the Bruhat order. Let ${}^J W$ denote the set of these minimal coset representatives. Each $w \in W$ factors uniquely as $w = vx$, $v \in W_J$, $x \in {}^J W$. We define the *hybrid basis* $\{CT_w\}$ of $\mathcal{H}$ by

$$CT_w = C_v T_x,$$

so it interpolates between the bases $\{T_w\}$, when $W_J = \{1\}$, and $\{C_w\}$, when $W_J = W$.

Theorem 3.2. *The matrix coefficients of right multiplication by C_w with respect to the hybrid basis, that is, the coefficients $c_{uv}^w(q)$ in the expansion*

$$CT_u\, C_w = \sum_v c_{uv}^w(q)\, CT_v,$$

are polynomials in $q^{\pm 1}$ with non-negative coefficients.

Theorem 3.2 generalizes theorems of Springer and Lusztig [19], [25] in the case $W_J = W$, and Dyer and Lehrer [4] in the case $W_J = \{1\}$ and $\dim(\mathfrak{g}) < \infty$. We give the proof in §§3.5–3.7, after reviewing the geometric interpretation of the Hecke algebra as a convolution algebra on the flag variety.

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[1]For simplicity, we write C_w for what would be $q^{l(w)/2} C'_w$ in the notation of [13].

3.3. Let $\mathcal{F}$ denote the (thin) flag variety of $\mathfrak{g}$ in the sense of Peterson-Kac [22] and Tits [27], [28]. We recall its construction and that of the associated Kac-Moody group G.

Fix an integrable highest-weight module $V = V(\lambda)$ of $\mathfrak{g}$ with λ regular. Let $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{h} \oplus \mathfrak{u}_+$ be the triangular decomposition, and set $\mathfrak{b} = \mathfrak{h} + \mathfrak{u}_+$. The algebraic torus $T = \exp(\mathfrak{h})$ acts diagonally on the weight spaces V_μ, and for each index i, the subalgebra $\langle e_i, h_i, f_i \rangle$ integrates to an action $\phi_i \colon SL_2 \to GL(V)$. Let $\widehat{\mathfrak{u}}_+ = \prod_{n \geq 1} (\mathfrak{u}_+)_n$ be the completion of $\mathfrak{u}_+$ with respect to the grading $\deg(e_i) = 1$. Given any $v \in V$, we have $(\mathfrak{u}_+)_{\geq n} v = 0$ for some n. Hence there is a well-defined exponential map from $\widehat{\mathfrak{u}}_+$ onto a subgroup $U \subseteq GL(V)$. We define $G \subseteq GL(V)$ to be the group generated by T, U and the $\phi_i(SL_2)$; N to be the subgroup generated by T and the elements $\tilde{s_i} = \phi_i(\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right])$, so $W = N/T$; and $B = T \ltimes U$. These form a Tits system with Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB.$$

For each $w \in W$, the space $D_w = \mathfrak{b} V_{w(\lambda)}$ is finite-dimensional and B-invariant, with B acting through a homomorphism $B \to B_w$ onto a solvable algebraic group $B_w \subseteq GL(D_w)$. Since $V_{w(\lambda)}$ is one-dimensional, it defines a distinguished, T-fixed point $e_w \in \mathbb{P}(D_w)$. The orbit $Y_w = Be_w$ is an affine *Schubert cell* of dimension $l(w)$, whose closure $X_w = \overline{Y_w}$ is a *Schubert variety.* For $v \leq w$, we have $D_v \subseteq D_w$, and these inclusions identify X_v with a closed subvariety of X_w. Then $X_w = \bigsqcup_{v \leq w} Y_v$ as a union of (locally closed) B-orbits. The flag variety is defined to be the ind-variety

$$\mathcal{F} = \varinjlim_{w} (X_w). \tag{2}$$

Different choices of λ yield the same varieties X_w, $\mathcal{F}$. The only thing that changes is the ample line bundle $\mathcal{O}(1)$ associated with the projective embedding $X_w \subseteq \mathbb{P}(D_w)$.

Note that the inductive system in (2) is filtered because every two elements $u, v \in W$ have an upper bound in the Bruhat order. In fact, there is a monoid structure $(W, *)$ on W such that $u * v = uv$ if $l(uv) = l(u) + l(v)$, and the Coxeter generators s_i are idempotent. Then $u, v \leq u * v$.

For $w > s_i w$, D_w and X_w are $\phi_i(SL_2)$-invariant, hence G acts on $\mathcal{F}$. The unique point $e_1 \in X_1 \subseteq \mathcal{F}$ has stabilizer equal to B, identifying $\mathcal{F}$ with G/B. Then $e_w = w(e_1)$ and $Y_w = BwB/B$. Let $G_u = \bigcup_{v \leq u} BvB$ be the preimage of X_u by $G \to G/B = \mathcal{F}$ and K_w the kernel of $B \to B_w$, for any $w \geq u$. Then G_u/K_w is naturally an algebraic variety and a principal B_w-bundle over X_u. We also have $G_u G_v \subseteq G_{u*v}$.

3.4. The canonical immersion $i \colon X_v \hookrightarrow X_w$ for $v \leq w$ induces an equivalence i_* of $\mathcal{D}_{\mathcal{M}}(X_v)$ onto the full subcategory of objects of $\mathcal{D}_{\mathcal{M}}(X_w)$ supported in X_v. We define $\mathcal{D}_{\mathcal{M}}(\mathcal{F}) = \varinjlim_w \mathcal{D}_{\mathcal{M}}(X_w)$. An object in $\mathcal{D}_{\mathcal{M}}(\mathcal{F})$ is *B-equivariant* if it is B_w-equivariant on any X_w containing its support. The category $\mathcal{D}_{\mathcal{M}}(\mathcal{F})$ is really just an organizing device for objects defined on Schubert varieties, so there is no need to invoke a more general theory of mixed Hodge modules on infinite-dimensional varieties. One defines $\mathcal{D}_{\mathcal{M}}(\mathcal{F} \times \mathcal{F})$ analogously.

The Grothendieck group $\mathcal{K}$ of B-equivariant Tate mixed Hodge modules on $\mathcal{F}$ is a free $\mathbb{Z}[q^{\pm 1}]$-module (§2.2) with two bases indexed by the elements of W, namely

$$\begin{aligned} C_w &= IC_{X_w}(\mathbb{Q}_{Y_w})[-l(w)], \\ T_w &= (i_{Y_w})_! \mathbb{Q}^H_{Y_w}, \end{aligned} \tag{3}$$

where $i_{Y_w}\colon Y_w \to \mathcal{F}$ is the inclusion. *Convolution* on $\mathcal{K}$ is defined in terms of the diagram

$$\begin{tikzcd}
\mathcal{F} & \mathcal{F}\times\mathcal{F} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathcal{F} \\
G \arrow[u, "q"] & G\times\mathcal{F} \arrow[l, "\pi_G"] \arrow[u, "r=q\times 1_{\mathcal{F}}"'] \arrow[d, "\wr"', "\eta\colon (g,hB)\mapsto(g,g^{-1}hB)"] & \\
& G\times\mathcal{F} \arrow[ul, "\pi_G"] \arrow[r, "\pi_{\mathcal{F}}"'] & \mathcal{F}\ ,
\end{tikzcd} \tag{4}$$

essentially by the formula

$$A_1 \cdot A_2 = (\pi_2)_!(\pi_1^*(A_1) \otimes r_\flat \eta^* \pi_{\mathcal{F}}^*(A_2)) = (\pi_2)_! r_\flat \eta^*(q^*(A_1) \boxtimes A_2),$$

where $r_\flat$ denotes B-equivariant descent. When $\dim(\mathfrak{g}) = \infty$, some care is required to make sense of this definition, since the individual tensor factors in the middle formula do not exist as objects of $\mathcal{D}_{\mathcal{M}}(\mathcal{F}\times\mathcal{F})$. The last formula, however, makes sense if interpreted in terms of the following approximation by finite-dimensional algebraic varieties: suppose A_1, A_2 supported in X_u, X_v, respectively, and let $w \geq u * v$. Replace the two factors $\mathcal{F}$ in the first two rows of (4) with X_u and X_w, the $\mathcal{F}$ in the bottom row with $X_{u^{-1}*w}$, and G with $G_u/K_{u^{-1}*w}$ (cf. §3.3). The resulting object $A_1 \cdot A_2$ is independent of the choices of u, v, w, and this construction defines an associative, $\mathbb{Z}[q^{\pm 1}]$-bilinear operation on the Grothendieck group of B-equivariant objects of $\mathcal{D}_{\mathcal{M}}(\mathcal{F})$. Note that the isomorphism of ind-schemes η becomes an immersion in the finite-dimensional approximation, so the functor η^* is still a weight-preserving equivalence (with inverse η_*) when restricted to objects supported in the image of η, among them $q^*(A_1) \boxtimes A_2$.

Since π_2 is proper and q is smooth, convolution commutes with the duality functor $\boldsymbol{D}$ and adds weights. One checks directly that the classes of the objects T_w in (3) satisfy the defining relations (1) of $\mathcal{H}$. This shows in particular that the convolution product preserves the Grothendieck group of Tate objects $\mathcal{K}$, and gives $\mathcal{K}$ the structure of an algebra which we can and will identify with $\mathcal{H}$. One also checks that $C_{s_i} = T_{s_i} + 1$ in $\mathcal{K}$ and $\boldsymbol{D}(C_w) = q^{-l(w)} C_w$, from which it follows that $\boldsymbol{D}$ corresponds to the involution $\bar{\cdot}$ of $\mathcal{H}$ and that the objects C_w satisfy the defining properties (i), (ii) of the Kazhdan-Lusztig basis in §3.1. In particular, $C_u \cdot C_v$ is pure of weight zero, hence (§2.2) its coefficients with respect to the basis C_w belong to $\mathbb{N}[q^{\pm 1}]$. This is the mixed Hodge module version of the proof of the theorem of Springer and Lusztig.

3.5. As in §3.1, fix $J \subseteq S$ and let $\mathfrak{l} \subseteq \mathfrak{g}$ be the Levi subalgebra generated by $\mathfrak{h}$ and $\{e_i, f_i : i \in J\}$, with Weyl group W_J. The Kac-Moody group of $\mathfrak{l}$ in the sense of §3.3 is the subgroup $L \subseteq G$ generated by T, $\exp(\widehat{\mathfrak{u}}_+(\mathfrak{l}))$, and the $\phi_i(SL_2)$ for $i \in J$. Let

$$P = LB = \bigcup_{w \in W_J} BwB$$

be the corresponding parabolic subgroup. The P-invariant subspace $V_L = \bigcup_{w \in W_J} D_w \subseteq V$ is an integrable highest-weight module for $\mathfrak{l}$. There is a semidirect decomposition $P = L \ltimes U_P$ such that P acts on V_L through the canonical homomorphism $P \to L$. We have $B_L = B \cap L$ and hence an isomorphism $\mathcal{F}_L \to P/B \subseteq \mathcal{F}$, $gB_L \mapsto gB$. More generally, if $x \in {}^J W$, the stabilizer ${}^x B$ of xB satisfies $B_L = {}^x B \cap L$, giving an isomorphism $i_x \colon \mathcal{F}_L \to LxB/B$, $gB_L \mapsto gxB$.

Let $\gamma \colon \mathbb{C}^* \to T$ be a dominant co-weight γ whose stabilizer in W is equal to W_J; then $\gamma(\mathbb{C}^*)$ centralizes L. The connected components of the fixed locus $\mathcal{F}^{\gamma(\mathbb{C}^*)}$ are the copies $i_x(\mathcal{F}_L) = LxB/B$ of $\mathcal{F}_L$, for $x \in {}^J W$ (this means that the intersections $X_w \cap i_x(\mathcal{F}_L)$ are the connected components of $X_w^{\gamma(\mathbb{C}^*)}$ for each w; in what follows, such interpretations will be left to the reader). The attracting variety to $i_x(\mathcal{F}_L)$ is $PxB/B = \bigcup_{v \in W_J} Y_{vx}$, with attracting map $\pi_x \colon PxB/B \to LxB/B$ induced by the homomorphism $P \to L$. Let $j_x \colon PxB/B \to \mathcal{F}$ be the inclusion.

3.6. The subalgebra $\mathcal{H}_J \subseteq \mathcal{H}$ spanned by $\{T_w \colon w \in W_J\}$ consists of the classes supported in $\mathcal{F}_L$ and and is naturally identified with the Hecke algebra of $\mathfrak{l}$. For $x \in {}^J W$, the map $\mathcal{H}_J \to \mathcal{H}$ induced by the functor $(j_x)_! \pi_x^* (i_x)_*$ is just convolution on the right by T_x, as one sees by applying it to the standard basis objects T_v, $v \in W_J$. In particular, we have $C_v T_x = (j_x)_! C$, where $C = \pi_x^* (i_x)_* C_v$ is an irreducible object pure of weight zero on PxB/B, since π_x is smooth.

Now, $\mathcal{F}$ is the disjoint union of the loci PxB/B for $x \in {}^J W$. Hence we deduce from the above description that for any $A \in \mathcal{H}$, the coefficient $\langle C_v T_x \rangle A$ of $C_v T_x$ in the expansion of A with respect to the hybrid basis is equal to $q^{-l(x)} \langle C_v \rangle i_x^* (\pi_x)_! j_x^* A$. The Tate twist $q^{-l(x)}$ appears because π_x has relative dimension $l(x)$.

3.7. Fix $u, v \in W_J$, $x, y \in {}^J W$, and $w \in W$. The object $A = r_\flat \eta^* (q^* (C_u T_y) \boxtimes C_w)$ on $\mathcal{F} \times \mathcal{F}$ is the extension by zero of a pure object A' of weight zero on $(PyB/B) \times \mathcal{F}$, and we have $C_u T_y C_w = (\pi_2)_! A = (\pi_2)_! A'$ (this last object is not pure, however, since PyB/B is not proper). For simplicity we write π_2 here and below for the second projection from any product. Define

$$j = (1 \times j_x) \colon (PyB/B) \times (PxB/B) \to (PyB/B) \times \mathcal{F},$$
$$\pi = (\pi_y \times \pi_x) \colon (PyB/B) \times (PxB/B) \to (LyB/B) \times (LxB/B).$$

Using base-change and the identity $\pi_2 \pi = \pi_x \pi_2$, we obtain $(\pi_x)_! j_x^* (\pi_2)_! A' = (\pi_2)_! \pi_! j^* A'$. Now $(LyB/B) \times (LxB/B)$ is a component of the fixed locus $((PyB/B) \times \mathcal{F})^{\gamma(\mathbb{C}^*)}$, with attracting variety $(PyB/B) \times (PxB/B)$ and attracting map π, and A' is equivariant for the diagonal action of T and hence of $\gamma(\mathbb{C}^*)$. Note that the defining projective embedding $X_w \hookrightarrow \mathbb{P}(D_w)$ of any Schubert variety is T-equivariant. By Proposition 2.4, therefore, $\pi_! j^* A'$ is pure of weight zero, and since LyB/B is proper, so is $(\pi_2)_! \pi_! j^* A'$. We have shown that $(\pi_x)_! j_x^* (C_u T_y C_w)$ is pure of weight zero. By §3.6 and §2.2, this implies that $\langle C_v T_x \rangle C_u T_y C_w \in \mathbb{N}[q^{\pm 1}]$, completing the proof of Theorem 3.2.

3.8. Fix notation as in §3.1 and let W^J be the set of elements $x \in W$ such that x is minimal in its *left* coset xW_J. Define

$$TC_w = T_x C_v,$$

where w factors uniquely as $w = xv$, with $x \in W^J$, $v \in W_J$. There is an anti-automorphism $\Psi(T_w) = T_{w^{-1}}$ of $\mathcal{H}$, which commutes with $\bar{\cdot}$ and hence satisfies $\Psi(C_w) = C_{w^{-1}}$. Applying Ψ yields the following equivalent form of Theorem 3.2, which is more convenient when one wants to regard $\mathcal{H}$ as a left rather than a right module over itself.

Corollary 3.9. *The coefficients $d^w_{vu}(q)$ in the expansion*

$$C_w \, TC_u = \sum_v d^w_{vu}(q) \, TC_v \tag{5}$$

are polynomials in $q^{\pm 1}$ with non-negative coefficients.

4. Application to parabolic Kazhdan-Lusztig polynomials

4.1. Fix $\mathcal{H}$, J, W, W_J and W^J as in §§3.1, 3.8, and $\mathcal{H}_J \subseteq \mathcal{H}$ as in §3.6. Let $\varepsilon(T_w) = (-1)^{l(w)}$ be the one-dimensional "sign" representation of $\mathcal{H}_J$. The induced representation $\operatorname{Ind}_{\mathcal{H}_J}^{\mathcal{H}}(\varepsilon)$ is then $\mathcal{H}e_J^-$, where the generator e_J^- satisfies

$$(T_{s_i} + 1)e_J^- = 0 \quad \text{for } i \in J. \tag{6}$$

The sets $\{T_w e_J^-\}$ and $\{C_w e_J^-\}$ form $\mathbb{Z}[q^{\pm 1}]$-bases of $\mathcal{H}e_J^-$ as w runs through W^J. The *parabolic Kazhdan-Lusztig polynomials* $P^{J-}_{vw}(q)$ of Deodhar [3] are the coefficients in the expansion

$$C_w e_J^- = \sum_v P^{J-}_{vw}(q) \, T_v e_J^-.$$

For $w \in W^J$ we have $TC_w e_J^- = T_w e_J^-$ by definition, while for $w \notin W^J$ we have $TC_w e_J^- = 0$ as a consequence of (6). In other words, $\mathcal{H}e_J^- \cong \mathcal{H}/\mathcal{N}$, where $\mathcal{N} = \operatorname{ann}(e_J^-)$ is spanned by $\{TC_w : w \notin W^J\}$. Hence P^{J-}_{vw} is equal to the coefficient of TC_v in the expansion (5) of $C_w \, TC_1$, yielding the following result.

Corollary 4.2. *The parabolic Kazhdan-Lusztig polynomials $P^{J-}_{vw}(q)$ have non-negative coefficients.*

When W_J is finite, Corollary 4.2 is a theorem of Kashiwara and Tanisaki [11].

4.3. Deodhar also defined *inverse* parabolic Kazhdan-Lusztig polynomials $Q^{J-}_{vw}(q)$, which up to a factor $(-1)^{l(w)-l(v)}$ are the coefficients in the expansion of $T_v e_J^-$ through the basis elements $C_w e_J^-$, as well as analogous polynomials P^{J+}_{vw}, Q^{J+}_{vw} for the induced representation $\mathcal{H}e_J^+$ such that $(T_{s_i} - q)e_J^+ = 0$ for $i \in J$. When W_J is finite, the theorem of Kashiwara and Tanisaki applies to all four variants. Of these, $P^{J-}_{vw}(q)$ and $Q^{J+}_{vw}(q)$ are the interesting ones, since $P^{J+}_{vw}(q)$ (for W_J finite) and $Q^{J-}_{vw}(q)$ (always) are equal to ordinary Kazhdan-Lusztig and inverse Kazhdan-Lusztig polynomials, respectively.

The positivity of P^{J+}_{vw} and Q^{J+}_{vw} remains an open problem for W_J infinite.

5. LLT polynomials associated to a reductive Lie group

We now apply the results of §3 in the special case that $\mathcal{H}$ is the affine Hecke algebra associated to a complex reductive group G. Using Theorem 3.2, we will deduce that certain formal q-characters of G, which can be naturally interpreted as *LLT polynomials associated to* G, have non-negative coefficients.

5.1. In what follows, we take the ground ring for Hecke algebras to be $\mathbb{Z}[u^{\pm 1}]$, with $\bar{\cdot}$ acting by $u \mapsto u^{-1}$,and define $q = u^2$ (any commutative ground ring and involutory automorphism such that $\overline{u} = u^{-1}$ would do, provided that $1 + q = 1 + u^2$ is not a zero-divisor). The traditional $\bar{\cdot}$-invariant Kazhdan-Lusztig basis is then $C'_w = u^{-l(w)} C_w$. The correspondingly renormalized standard basis will be denoted $T'_w = u^{-l(w)} T_w$.

5.2. Fix the Cartan data specifying a complex reductive Lie group G, *i.e.*, the weight and co-weight lattices X, $X^\vee$ and the sets of simple roots and co-roots $\alpha_i \in X$, $\alpha_i^\vee \in X^\vee$, such that $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ is a Cartan matrix of finite type. Let $(W_f, S_f = \{s_1, \ldots, s_n\})$ be the Weyl group of G, acting on X. The *extended affine Weyl group* is the semidirect product

$$W = W_f \ltimes X. \tag{7}$$

We write $\tau(\lambda)$ for the element of W corresponding to translation by $\lambda \in X$.

Let $Q \subseteq X$ be the root lattice and let θ be the dominant short root (so $\theta^\vee$ is the highest co-root). The unextended affine Weyl group $W_a = W_f \ltimes Q$ is the Weyl group of the affine Kac-Moody algebra $\widehat{\mathfrak{g}_l}$, where $\mathfrak{g}_l$ is Langlands dual to $\mathfrak{g} = \mathrm{Lie}(G)$. It is generated as a Coxeter group by S_f and the reflection $s_0 = \tau(\theta) s_\theta$ through the hyperplane $\langle \alpha_0^\vee, \lambda \rangle = 0$, where $\alpha_0^\vee$ is the affine-linear function $\langle \alpha_0^\vee, \lambda \rangle = \langle \theta^\vee, \lambda \rangle - 1$. The *basic alcove* is the fundamental domain

$$\mathcal{A}_0 = \{x : \langle \alpha_i^\vee, x \rangle \geq 0 \text{ for all } i = 0, 1, \ldots, n\} \subseteq X_\mathbb{R} = X \otimes_\mathbb{Z} \mathbb{R}$$

for the action of W_a on $X_\mathbb{R}$. The stabilizer Π of $\mathcal{A}_0$ in W acts on W_a, permuting the simple reflections (which correspond to the walls of $\mathcal{A}_0$). We have $W = \Pi \ltimes W_a$ and $\Pi \cong X/Q$. One defines $l(w) = l_{W_a}(v)$, where $w = \pi v$ with $\pi \in \Pi$, $v \in W_a$.

The *extended affine Hecke algebra* $\mathcal{H}$ is the twisted group algebra $\Pi \cdot \mathcal{H}(W_a)$ with multiplication law

$$(\pi a)(\pi' b) = \pi\pi' \cdot \pi'^{-1}(a) b.$$

It has a *standard basis* $T_w = \pi T_v$ and a *Kazhdan-Lusztig basis* $C_w = \pi C_v$, where again $w = \pi v$. Extending the involution $\bar{\cdot}$ of $\mathcal{H}(W_a)$ to $\mathcal{H}$ by the rule $\overline{\pi} = \pi$, the Kazhdan-Lusztig basis is characterized by the same properties as in §3.1.

Fix $J \subseteq S_f$ and the parabolic subgroup $W_J \subseteq W_f \subseteq W$, and define the extended hybrid basis $\{TC_w = \pi TC_v\}$ of $\mathcal{H}$, where $\{TC_v\}$ is the basis of $\mathcal{H}(W_a)$ from §3.8, and $w = \pi v$ as before. Then Corollary 3.9 is also valid in the extended setting, since we have

$$(\pi C_w)(\pi' TC_u) = \sum_{v \in W_a} d_{vu}^{\pi'^{-1}(w)}(q)\, \pi\pi' TC_v$$

with the same coefficients as in (5).

5.3. We recall Bernstein's presentation of the affine Hecke algebra $\mathcal{H}$ and description of its center [14, (4.4)]. Let $X_+ \subseteq X$ be the cone of dominant weights. For $\lambda \in X_+$, define $Y^\lambda = T'_{\tau(\lambda)}$, and extend the definition to all $\lambda \in X$ by setting $Y^\lambda = Y^\mu (Y^\nu)^{-1}$, where $\lambda = \mu - \nu$ with $\mu, \nu \in X_+$. The Y^λ are well-defined and satisfy

$$Y^\lambda Y^\mu = Y^{\lambda+\mu} \tag{8}$$

for all $\lambda, \mu \in X$, because (8) holds for $\lambda, \mu \in X_+$ to begin with. The elements $Y^\lambda T'_w$ for $\lambda \in X$, $w \in W_f$ form a basis of $\mathcal{H}$. The multiplication law is given in terms of this basis by (8), the multiplication in $\mathcal{H}(W_f)$, and the relations

$$T'_{s_i} Y^\lambda - Y^{s_i(\lambda)} T'_{s_i} = (u - u^{-1}) \frac{Y^\lambda - Y^{s_i(\lambda)}}{1 - Y^{-\alpha_i}} \quad (i \neq 0). \tag{9}$$

The commutative subalgebra $\mathcal{Y} = \mathbb{Z}[u^{\pm 1}] Y^X \subseteq \mathcal{H}$ inherits an action of W_f from its identification with the group algebra $\mathbb{Z}[u^{\pm 1}]X$.

Proposition 5.4. *The center of $\mathcal{H}$ is the subalgebra of W_f-invariants $Z(\mathcal{H}) = \mathcal{Y}^{W_f}$.*

The characters χ_λ of G belong to $(\mathbb{Z}X)^{W_f} \subseteq (\mathbb{Z}[u^{\pm 1}]X)^{W_f} \cong \mathcal{Y}^{W_f} = Z(\mathcal{H})$. In what follows, we identify χ_λ with an element of $Z(\mathcal{H})$. Let w_0 be the longest element of W_f, and define

$$e^+ = C'_{w_0} = u^{-l(w_0)} \sum_{w \in W_f} T_w.$$

The elements $w_0 \tau(\lambda)$, $\lambda \in X_+$ are the maximal representatives of the double cosets $W_f w W_f$, hence the Kazhdan-Lusztig basis elements $C'_{w_0 \tau(\lambda)}$ form a basis of $e^+ \mathcal{H} e^+$.

Proposition 5.5 (Lusztig [18])**.** *There holds the identity $e^+ \chi_\lambda = \chi_\lambda e^+ = C'_{w_0 \tau(\lambda)}$.*

The following corollary will be used in §6 and §7.

Corollary 5.6. *Regarded as elements of $Z(\mathcal{H})$, the characters χ_λ of G satisfy $\chi_\lambda = \overline{\chi_\lambda}$.*

Proof. Proposition 5.5 implies that there is a linear isomorphism $\phi \colon Z(\mathcal{H}) \to e^+ \mathcal{H} e^+$ given by $\phi(\chi) = e^+ \chi$. Since $e^+ = \overline{e^+}$, ϕ commutes with $\bar{\cdot}$. By Proposition 5.5, $\phi(\chi_\lambda) = \overline{\phi(\chi_\lambda)}$, hence $\chi_\lambda = \overline{\chi_\lambda}$. $\square$

5.7. Let w_0^J be the longest element of W_J and define

$$e_J^- = u^{-l(w_0^J)} \sum_{w \in W_J} (-q)^{l(w_0^J) - l(w)} T_w. \tag{10}$$

Equivalently, e_J^- is the signed Kazhdan-Lusztig basis element denoted $C_{w_0^J}$ in the traditional notation of [13]. It has the property that $\mathbb{Z}[u^{\pm 1}] e_J^-$ is a two-sided $\mathcal{H}_J$-submodule of $\mathcal{H}$, on which $\mathcal{H}_J$ acts (on either side) by the sign representation ε. We have $\mathcal{H} e_J^- \cong \operatorname{Ind}_{\mathcal{H}_J}^{\mathcal{H}}(\varepsilon)$, making the notation in §4 consistent with (10).

The space $e_J^-\mathcal{H}e^+$ is a $Z(\mathcal{H})$-module. In order to describe a basis of this module, we introduce the following terminology. A double coset $W_J\,wW_f$ is *regular* if it is a regular orbit of the (left)$\times$(right) action of $W_J \times W_f$ on W, or equivalently, if the orbit of wW_f (resp. $W_J\,w$) is regular for the left action of W_J on W/W_f (resp. the right action of W_f on $W_J\backslash W$).

Lemma 5.8. *(i) If $W_J\,wW_f$ is not regular, then $e_J^-T_we^+ = 0$.*

(ii) If $W_J\,wW_f$ is regular, v is its minimal element, and $w = xvy$ for (unique) $x \in W_J$, $y \in W_f$, then $e_J^-T_we^+ = (-1)^{l(x)}q^{l(y)}e_J^-T_ve^+$.

(iii) As w ranges over the minimal representatives of regular double cosets $W_J\,wW_f$, the elements $e_J^-T_we^+$ form a basis of $e_J^-\mathcal{H}e^+$.

Proof. (i) We may assume that w is minimal in $W_J\,wW_f$. The non-regularity implies that there exist indices $j \in J$ and $1 \le i \le n$ such that $s_jw = ws_i$, and hence $T_{s_j}T_w = T_wT_{s_i}$. Acting on left and right by e_J^-, e^+ gives $-e_J^-T_we^+ = qe_J^-T_we^+$, hence $e_J^-T_we^+ = 0$.

(ii) The factorization $w = xvy$ is reduced, $e_J^-T_x = (-1)^{l(x)}e_J^-$, and $T_ye^+ = q^{l(y)}e^+$.

(iii) It follows from (i) and (ii) that the set in question spans $e_J^-\mathcal{H}e^+$. For indices w belonging to distinct double cosets, it is clear that the elements $e_J^-T_we^+$ are independent. $\square$

The basis described in Lemma 5.8 (iii) is the *standard basis* of the module $e_J^-\mathcal{H}e^+$.

Theorem 5.9. *The matrix coefficients of the operator $\chi_\lambda \in Z(\mathcal{H})$ with respect to the standard basis of $e_J^-\mathcal{H}e^+$, that is, the coefficients in the expansion*

$$\chi_\lambda\, e_J^-T_we^+ = \sum_v P_{vw}^\lambda(u)\, e_J^-T_ve^+, \tag{11}$$

have the property that $u^lP_{vw}^\lambda(u)$ is a polynomial in $q^{\pm 1}$ with non-negative coefficients, for some integer l.

Before giving the proof, we remark that $\Psi(e^+) = e^+$, $\Psi(e_J^-) = e_J^-$, where Ψ is the anti-automorphism of $\mathcal{H}$ defined in §3.8. Using Proposition 5.5 and the fact that $(w_0\tau(\lambda))^{-1} = w_0\tau(w_0(-\lambda))$, we also have $\Psi(\chi_\lambda) = \chi_{w_0(-\lambda)} = \chi_\lambda^*$, the contragredient of χ_λ. Applying Ψ to (11) therefore yields an equivalent expansion

$$\chi_\lambda\, e^+T_we_J^- = \sum_v P_{v^{-1}w^{-1}}^{w_0(-\lambda)}(u)\, e^+T_ve_J^-, \tag{12}$$

where v and w range over minimal representatives of regular double cosets $W_f\,wW_J$.

Proof of Theorem 5.9. It is equivalent to prove the same thing for the coefficients in (12). Regard $e^+\mathcal{H}e_J^-$ as a submodule of $\mathcal{H}e_J^-$. For the latter, take the basis $\{T_we_J^- : w \in W^J\}$ from §4. For v minimal in a regular double coset $W_f\,vW_J$, the basis element $T_ve_J^-$ of $\mathcal{H}e_J^-$ occurs with coefficient $u^{-l(w_0)}$ in $e^+T_ve_J^-$, and with zero coefficient in every other standard basis element $e^+T_we_J^-$ of $e^+\mathcal{H}e_J^-$. Letting $\langle - \rangle$ stand for taking a coefficient, we therefore have

$$\langle e^+T_ve_J^-\rangle\chi_\lambda\, e^+T_we_J^- = u^{l(w_0)}\langle T_ve_J^-\rangle\chi_\lambda\, e^+T_we_J^- = u^{l(w_0)-l(w_0\tau(\lambda))}\langle T_ve_J^-\rangle C_{w_0\tau(\lambda)}T_we_J^-.$$

By the same reasoning as in §4, the last expression is equal to $u^{l(w_0)-l(w_0\tau(\lambda))}\langle TC_v\rangle C_{w_0\tau(\lambda)}TC_w$, and hence has the desired form by Corollary 3.9. $\square$

Remark 5.10. We can make the factor u^l in the conclusion of Theorem 5.9 more explicit, as follows. The elements of $X \cap \mathcal{A}_0$ are called *minuscule weights*. Given $\lambda \in X$, define

$$l_\lambda = l(\tau(\lambda_\pi)),$$

where λ_π is the unique minuscule weight in the coset $\lambda + Q$. Let p be the composite of the canonical homomorphisms $W \to W/W_a \cong \Pi$ and $\Pi \hookrightarrow W \to W/X \cong W_f$. Then $l_\lambda = l(p(\tau(\lambda_\pi))) = l(p(\tau(\lambda)))$.

Let $2\rho^\vee$ be the sum of the positive co-roots. For $\lambda \in X_+$ dominant, $l(\tau(\lambda))$, which is the number of affine hyperplanes separating $\mathcal{A}_0$ from $\mathcal{A}_0 + \lambda$, is equal to $\langle 2\rho^\vee, \lambda\rangle$. Hence $l(\tau(\lambda)) - l_\lambda$ is even, since λ, λ_π are both dominant and $\lambda - \lambda_\pi \in Q$. For $\lambda \in X_+$ we also have that the product $w_0\tau(\lambda)$ is reduced, giving $l(w_0) - l(w_0\tau(\lambda)) = -l(\tau(\lambda))$. The proof of Theorem 5.9 then shows that $u^{l_\lambda} P^\lambda_{vw}(u) \in \mathbb{Z}[q^{\pm 1}]$. Now, the twisted group algebra structure $\mathcal{H} = \Pi \cdot \mathcal{H}(W_a)$ implies that if $P^\lambda_{vw}(u) \neq 0$, then $p(\tau(\lambda)) = p(vw^{-1})$. Given v and w, we therefore see that $u^{l(p(vw^{-1}))} P^\lambda_{vw}(u)$ is a polynomial in $q^{\pm 1}$ for all λ.

Remark 5.11. Let $X_{++}(L) = \{\lambda \in X : \langle \alpha_i^\vee, \lambda\rangle > 0 \ \forall i \in J\}$ denote the set of weights regular and dominant for the Levi subgroup $L \subseteq G$ whose Weyl group is W_J. The definition (7) canonically identifies W/W_f with X and hence $W_J\backslash W/W_f$ with $W_J\backslash X$. Regular double cosets correspond to regular W_J-orbits in X, each of which contains a unique element of $X_{++}(L)$. Thus there is a canonical bijection

$$X_{++}(L) \to (W_J\backslash W/W_f)_{\mathrm{reg}}, \quad \lambda \mapsto W_J\,\tau(\lambda)W_f. \tag{13}$$

By a *formal q-character of G* we shall mean a formal linear combination (typically infinite) of the characters χ_λ, with coefficients in $\mathbb{Z}[q^{\pm 1}]$. Given $\beta, \gamma \in X_{++}(L)$, let $v \in W_J\,\tau(\beta)W_f$, $w \in W_J\,\tau(\gamma)W_f$ be the minimal representatives. Fixing v and w, we collect the coefficients $P^\lambda_{vw}(u)$ in (11) for varying λ into a generating function:

$$\mathcal{L}^G_{L,\beta,\gamma}(q) = u^{l_{\beta-\gamma}} \sum_\lambda P^\lambda_{vw}(u)\chi_\lambda.$$

Since $l_{\beta-\gamma} = l(p(vw^{-1}))$, Remark 5.10 shows that $\mathcal{L}^G_{L,\beta,\gamma}(q)$ is a formal q-character.

Definition 5.12. The formal q-characters $\mathcal{L}^G_{L,\beta,\gamma}(q)$ are *LLT polynomials associated to G* (this terminology will be justified in §6).

Remark 5.13. Specializing at $u = 1$ identifies $\mathcal{H}e^+ = \mathcal{Y}e^+$ with the group algebra $\mathbb{Z}X$. We denote $\lambda \in X$ by the multiplicative notation y^λ when regarded as an element of $\mathbb{Z}X$. Fix $\rho_L \in X$ such that $\langle \alpha_i^\vee, \rho_L\rangle = 1$ for $i \in J$, so $X_{++}(L) = X_+(L) + \rho_L$. Given $\beta \in X_+(L)$, let v be the minimal representative of $W_J\,\tau(\beta + \rho_L)W_f$. Then $e_J^- T_v e^+$ specializes at $u = 1$ to

$$a_{\beta+\rho_L} \underset{\mathrm{def}}{=} \sum_{w \in W_J} (-1)^{l(w_0^J) - l(w)} y^{w(\beta+\rho_L)} = \chi_\beta(L) a_{\rho_L}.$$

It follows that the coefficient $P^\lambda_{vw}(1)$ of χ_λ in $\mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(1)$ is equal to the multiplicity of $\chi_\beta(L)$ in $\chi_\gamma(L) \otimes (\chi_\lambda|L)$, a fact which can be interpreted as an identity of formal characters

$$\mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(1) = \mathrm{Ind}_L^G(\chi_\beta(L) \otimes \chi_\gamma(L)^*).$$

We should point out that for $\beta, \gamma \in X_+(L)$ fixed, *the q-character* $\mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(q)$ *depends on the choice of* ρ_L, even though $\mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(1)$ does not. In §7, we will see how to express $\mathcal{L}^G_{L,\beta,\gamma}(q)$ as a finite linear combination, depending on ρ_L, of certain other formal induced q-characters $\mathrm{Ind}^G_{L,q^{-1}}(\chi_\mu(L))$ for $\mu \in X_+(L)$, whose definition does not involve ρ_L.

6. Comparison with combinatorial LLT polynomials

In this section we show that the LLT polynomials $\mathcal{L}^G_{L,\beta,\gamma}(q)$ for $G = GL_n$ coincide (after minor adjustments) with the polynomials defined combinatorially by Lascoux, Leclerc and Thibon (LL&T) [16]. In view of Theorem 5.9, it follows that the coefficients in the Schur polynomial expansion of any combinatorial LLT polynomial are positive, as had been proven in a special case by Leclerc and Thibon [17] and conjectured in general. Most of the ideas used below can already be found in [16], [17], but we provide a self-contained exposition for the reader's convenience and in order to bring out those aspects which also apply to groups other than GL_n.

6.1. We shall describe a procedure (Corollary 6.4 and the discussion thereafter) for computing the coefficients $P^\lambda_{vw}(u)$ in (11). To do this, we first need an alternative system for indexing double cosets $W_f\, wW_J$. The notation here and below is the same as in §5.2.

Let $\eta \in -X_+$ be an anti-dominant weight such that $\mathrm{Stab}^{W_f}(\eta) = W_J$ and let k be an integer such that

(a) $k > -\langle \theta^\vee, \eta\rangle$;
(b) $W_f\, \eta \cap (\eta + kX) = \{\eta\}$.

Since $W_f\, \eta$ is finite, any sufficiently large k satisfies these conditions. We indicate with a raised dot the *level* $-k$ *action* of W on X, in which W_f acts as usual, but translations are dilated so that $\tau(\lambda)\cdot\mu = \mu - k\lambda$. Condition (b) ensures that $\mathrm{Stab}^W(\eta) = W_J$ for this action. Identify the coset space W/W_J with the orbit $W\cdot\eta$. Then $W_f\backslash W/W_J$ is identified with the set of W_f-orbits in $W\cdot\eta$, regular double cosets correspond to regular orbits, and we get a bijection

$$X_{++} \cap (W\cdot\nu) \to (W_f\backslash W/W_J)_{\mathrm{reg}}, \qquad \mu = w(\eta + k\beta) \mapsto W_f\, \tau(-\beta)W_J.$$

The expression $\mu = w(\eta + k\beta) = \tau(w(-\beta))w\cdot\eta$ is unique if we require $w \in W^J$.

This bijection is related to the bijection $X_{++}(L) \to (W_J\backslash W/W_f)_{\mathrm{reg}}$ in (13) as follows. Since $\mu \in X_{++}$, $w \in W^J$, and $\langle \alpha_i^\vee, \eta\rangle = 0$ for all $i \in J$, we see that $\beta \in X_{++}(L)$. Composing our two bijections with the canonical bijection $(W_f\backslash W/W_J)_{\mathrm{reg}} \cong (W_J\backslash W/W_f)_{\mathrm{reg}}$ given by $W_f\, wW_J \leftrightarrow W_J\, w^{-1}W_f$, we have

$$\mu \leftrightarrow W_f\, \tau(-\beta)W_J \leftrightarrow W_J\, \tau(\beta)W_f \leftrightarrow \beta.$$

6.2. Fix W_J, η and k as in §6.1. Given $\mu = w(\eta + k\beta) \in W\cdot\eta$, where $w \in W^J$, set $\lambda = w(-\beta)$, so $\mu = \tau(\lambda)w\cdot\eta$, and define

$$V(\mu) = \overline{Y^\lambda} T'_w e^-_J \in \mathcal{H}e^-_J, \qquad |\mu\rangle = e^+V(\mu) \in e^+\mathcal{H}e^-_J.$$

It is clear from the Bernstein presentation (§5.3) that the $V(\mu)$ form a basis of $\mathcal{H}e_J^-$. Their images $|\mu\rangle$ span $e^+\mathcal{H}e_J^-$ and satisfy linear relations described by the following proposition.

Proposition 6.3. *(i) For $\mu = w(\eta+k\beta) \in X_{++}\cap(W\cdot\eta)$, where $w \in W^J$, and v the minimal element of $W_f\,\tau(-\beta)W_J$, we have $\tau(w(-\beta))w = w_0 v$ and $|\mu\rangle = u^{l(w_0)}e^+T'_v e_J^-$.*

(ii) The elements $|\mu\rangle$ for all $\mu \in W\cdot\eta$ satisfy the following straightening relations. *Assume $\langle\alpha_i^\vee,\mu\rangle \leq 0$ for some $i \neq 0$ and set $-\langle\alpha_i^\vee,\mu\rangle = pk + r$, where $0 \leq r < k$. Then*

$$\begin{cases} |\mu\rangle = 0 & \text{if } p = r = 0, \text{ i.e., if } s_i\mu = \mu \\ |\mu\rangle = -|s_i\mu\rangle & \text{if } r = 0 \text{ and } p > 0 \\ |\mu\rangle = u^{-1}|s_i\mu\rangle & \text{if } r \neq 0 \text{ and } p = 0 \\ |\mu\rangle = u^{-1}|s_i\mu\rangle + u^{-1}|\mu + r\alpha_i\rangle - |s_i\mu - r\alpha_i\rangle & \text{if } r \neq 0 \text{ and } p > 0. \end{cases} \tag{14}$$

Proof. (i) Set $\lambda = w(-\beta)$. Condition (a) in §6.1 implies that $\langle\alpha_i^\vee, w(\eta)\rangle < k$ for all i, and since $\mu = w(\eta) - k\lambda$ is dominant, it follows that $\lambda \in -X_+$. The product $\tau(\lambda)w$ is therefore reduced for all $w \in W_f$, and consequently $\tau(\lambda)w$ is minimal in $\tau(\lambda)wW_J$ for $w \in W^J$. Condition (a) in §6.1 also implies that $-\eta/k \in \mathcal{A}_0$, hence $-\mu/k \in \tau(\lambda)w(\mathcal{A}_0)$. Now $-\mu/k$ is an interior point of $-(X_\mathbb{R})_+$, so $\tau(\lambda)w(\mathcal{A}_0) \subseteq -(X_\mathbb{R})_+$, and therefore $\tau(\lambda)w$ is maximal in $W_f\,\tau(\lambda)w$ as well as minimal in $\tau(\lambda)wW_J$. Equivalently, $\tau(\lambda)w = w_0 v$, where v is the minimal element in $W_f\,\tau(\lambda)wW_J = W_f\,\tau(-\beta)W_J$, as claimed. Moreover, since $-\lambda \in X_+$, we have $\overline{Y^\lambda} = \overline{T'^{-1}_{\tau(-\lambda)}} = T'_{\tau(\lambda)}$. In the equation $\tau(\lambda)w = w_0 v$, each side is a reduced product, hence $V(\mu) = T'_{\tau(\lambda)w}e_J^- = T'_{w_0}T'_v e_J^-$ and $|\mu\rangle = e^+T'_{w_0}T'_v e_J^- = u^{l(w_0)}e^+T'_v e_J^-$.

(ii) Using (9) one verifies that T'_{s_i} commutes with $\overline{Y^\lambda}+\overline{Y^{s_i(\lambda)}}$ and that $T'_{s_i}(\overline{Y^\lambda}+\overline{Y^{s_i(\lambda)+\alpha_i}}) = (\overline{Y^{s_i(\lambda)}} + \overline{Y^{\lambda-\alpha_i}})T'^{-1}_{s_i}$ for all $\lambda \in X$. Since $e^+(T'_{s_i} - u) = 0$, this yields the identities

$$e^+(\overline{Y^\lambda} + \overline{Y^{s_i(\lambda)}})(T'_{s_i} - u)T'_w e_J^- = 0 \tag{15}$$

$$e^+\big((\overline{Y^{s_i(\lambda)}} + \overline{Y^{\lambda-\alpha_i}})T'^{-1}_{s_i} - u(\overline{Y^\lambda} + \overline{Y^{s_i(\lambda)+\alpha_i}})\big)T'_w e_J^- = 0. \tag{16}$$

The straightening relations follow by applying (15) for $\mu = w(\eta) - k\lambda$ such that $s_i w > w$, and (16) for μ such that $s_i w < w$. ☐

For $G = GL_n$, Proposition 6.3 is [17, Propositions 5.2, 5.9]. We alert the reader that our u is $-q$ in [17], our $\overline{Y^\beta}$ is Y^β, and our $\tau(\lambda)$ is y^λ.

Using (12) and Proposition 6.3(i), we get the following expression for $P^\lambda_{vw}(u)$.

Corollary 6.4. *Fix W_J, η and k as in §6.1.*

(i) The elements $|\mu\rangle$ for $\mu \in X_{++}\cap(W\cdot\eta)$ form a basis of $e^+\mathcal{H}e_J^-$.

(ii) Let v, w be the minimal elements of regular double cosets $W_J\,vW_f$, $W_J\,wW_f$, and set $\mu = w_0v^{-1}\cdot\eta$, $\nu = w_0w^{-1}\cdot\eta$. Then μ and ν belong to X_{++}, and the coefficients in (11) are given by

$$P^\lambda_{vw}(u) = u^{l(w)-l(v)}Q^\lambda_{\mu\nu}(u), \quad \textit{where} \quad \chi^*_\lambda\,|\nu\rangle = \sum_{\mu\in X_{++}\cap(W\cdot\eta)} Q^\lambda_{\mu\nu}(u)\,|\mu\rangle.$$

To see how this is useful, note that if $\nu \in W\cdot\eta$ is not in X_{++}, the straightening relations (14) express $|\nu\rangle$ in terms of elements $|\mu\rangle$ such that $\mu < \nu$ in the partial ordering on $X = W/W_f$ induced by the Bruhat order on W. Given $\chi = \sum a_\lambda Y^\lambda \in Z(\mathcal{H}) = \mathcal{Y}^{W_f}$ with coefficients $a_\lambda \in \mathbb{Z}$, we have $\chi = \overline{\chi} = \sum a_\lambda \overline{Y^\lambda}$ by Corollary 5.6, and hence

$$\chi\,|\nu\rangle = \sum a_\lambda\,|\nu - k\lambda\rangle. \tag{17}$$

The right-hand side can now be expressed in terms of the basis $\{|\mu\rangle : \mu \in X_{++} \cap (W\cdot\eta)\}$ by applying (14) recursively. When $\chi = \chi^*_\lambda$, this computes the coefficients $Q^\lambda_{\mu\nu}$ in Corollary 6.4.

6.5. We now recall the combinatorially defined LLT polynomials from [16], [17]. A *k-ribbon* is a connected skew Young diagram of size k, containing no 2×2 square. The *spin* of a k-ribbon is the number of rows in it, less one. A *horizontal k-ribbon strip* is a skew shape κ which can be tiled by k-ribbons in such a way that the bottom-right box in each ribbon is the bottom box in a column of κ (we picture Young diagrams in the French orientation, with the largest part of a partition in the bottom row of its diagram). The spin of κ is the sum of the spins of the ribbons in its tiling, which is unique. A *semistandard k-ribbon tableau* of shape μ/ν is a sequence of skew diagrams

$$\nu = \nu_0 \subseteq \nu_1 \subseteq \nu_2 \subseteq \cdots \subseteq \mu$$

such that each ν_i/ν_{i-1} is a (possibly empty) horizontal k-ribbon strip and $\nu_i = \mu$ for some i. One thinks of the ribbons in the tiling of ν_i/ν_{i-1} as labelled with the integer i. The set of semistandard k-ribbon tableaux of shape μ/ν is denoted $\mathrm{SSRT}_k(\mu/\nu)$. Given $T \in \mathrm{SSRT}_k(\mu/\nu)$, define

$$\mathrm{spin}(T) = \sum_i \mathrm{spin}(\nu_i/\nu_{i-1}),$$
$$x^T = x_1^{\beta_1} x_2^{\beta_2}\cdots x_l^{\beta_l} \quad \text{where } |\nu_i/\nu_{i-1}| = k\beta_i.$$

For examples, see [17, Figs. 2–5].

Definition 6.6. *Combinatorial LLT polynomials* are the ribbon tableau generating functions

$$G^{(k)}_{\mu/\nu}(x;u) = \sum_{T\in \mathrm{SSRT}_k(\mu/\nu)} u^{\mathrm{spin}(T)} x^T.$$

6.7. Take $G = GL_n$, $X = \mathbb{Z}^n$, so $W_f = S_n$ acts by permuting coordinates. The dominant weights λ such that $\lambda_n \geq 0$ are just integer partitions $(\lambda_1 \geq \cdots \geq \lambda_n)$ with possible trailing zeroes. Their corresponding characters χ_λ are the *polynomial characters*.

Let $W_J \subseteq W_f = S_n$ be a standard parabolic subgroup, *i.e.*, a Young subgroup $S_{r_1}\times\cdots\times S_{r_l}$, where $J = \{r_1, r_1+r_2, \ldots, n-r_l\}$. Suppose $\eta \in X$ and $k \in \mathbb{Z}_+$ are such that

$$-k < \eta_1 = \cdots = \eta_{r_1} < \eta_{r_1+1} = \cdots = \eta_{r_1+r_2} < \cdots < \eta_{n-r_l+1} = \ldots = \eta_n \leq 0. \tag{18}$$

Then η, k and W_J satisfy the conditions in §6.1. Moreover, given k, every weight μ belongs to $W\cdot\eta$ for some η satisfying (18)—namely, reduce $\mu \pmod k$ to a vector with entries in $[1-k, 0]$ and sort this into increasing order to get η.

Define $\rho_n = (0, -1, \ldots, 1-n)$. If $\nu \subseteq \mu$ are partitions with at most n parts such that μ/ν is a k-ribbon, then $\mu + \rho_n$ is a permutation of $\nu + \rho_n + k\epsilon_i$ for some unit vector ϵ_i. It follows more generally that if μ/ν can be tiled by k-ribbons, then $\mu + \rho_n$ and $\nu + \rho_n$ belong to the same orbit $W \cdot \eta$.

Proposition 6.8. *Let $\nu \subseteq \mu$ be partitions with at most n parts such that μ/ν can be tiled by k-ribbons. Put $\mu + \rho_n = v(\eta + k\beta)$, $\nu + \rho_n = w(\eta + k\gamma)$, with η as in* (18) *and $v, w \in W^J$, where $W_J = \operatorname{Stab}^{S_n}(\eta)$ (if $W_J = S_{r_1} \times \cdots \times S_{r_l}$, this gives $\beta, \gamma \in X_{++}(L)$, where $L = GL_{r_1} \times \cdots \times GL_{r_l}$). Setting $q = u^2$ as always, we have*

$$G^{(k)}_{\mu/\nu}(x; u^{-1}) = u^{l(v\tau(-\beta)) - l(w\tau(-\gamma)) - l_{\beta-\gamma}} \mathcal{L}^{GL_n}_{L,\beta,\gamma}(q)_{\mathrm{pol}}, \tag{19}$$

where $(\sum_\lambda a_\lambda \chi_\lambda)_{\mathrm{pol}} = \sum_{\lambda_n \geq 0} a_\lambda \chi_\lambda$ denotes the truncation of a formal GL_n character to polynomial characters, and the polynomial character χ_λ is identified with the Schur polynomial $s_\lambda(x)$.

Proposition 6.8 will be proven in §6.13. In the statement, x stands for an infinite alphabet $x_1, x_2, \ldots$, as it does also in the definition of $G^{(k)}_{\mu/\nu}(x; u)$. Although it is more natural to identify χ_λ with the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ in n variables, the distinction is unimportant, since in fact $G^{(k)}_{\mu/\nu}(x, u)$ belongs to the vector space spanned by Schur polynomials $s_\lambda(x)$ such that λ is a partition with at most n parts (this follows either from Proposition 6.8 itself or from Lemma 6.12).

Corollary 6.9. *The coefficients in the Schur polynomial expansion*

$$u^{-m} G^{(k)}_{\mu/\nu}(x; u) = \sum_\lambda g^{\lambda,k}_{\mu/\nu}(q) s_\lambda(x),$$

where $m = \min_{T \in \mathrm{SSRT}_k(\mu/\nu)}(\operatorname{spin}(T))$, are polynomials in q with non-negative coefficients.

This was conjectured in [16] and proved in [17, Theorem 4.1] in the case when ν is empty, and more generally when ν is a k-core [7, Proposition 3.5.1].

Corollary 6.10. *The coefficients $\tilde{K}_{\lambda\mu}(q, t)$ in the Schur polynomial expansion of the transformed Macdonald polynomials (see, e.g.,* [8, §2] *for definitions)*

$$\tilde{H}_\mu(x; q, t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q, t) s_\lambda(x)$$

are polynomials in q and t with non-negative integer coefficients.

This follows from Corollary 6.9 and the expression for $\tilde{H}_\mu(x; q, t)$ in terms of LLT polynomials given in [6, Theorem 2.2, Proposition 3.4, and eq. (23)]. It was first proved in [8] using a different method based on the geometry of Hilbert schemes.

6.11. As in [21], let ω be the involutory automorphism of the algebra of symmetric functions in infinitely many variables such that $\omega(e_r) = h_r$ and $\omega(s_\lambda) = s_{\lambda'}$, where e_r and h_r denote, respectively, the elementary and complete homogeneous symmetric functions of degree r, and λ' denotes the transpose of a partition λ. The argument in the proof of [17, Theorem 6.4] also proves the following lemma.

Lemma 6.12. *There holds the identity*

$$\omega G^{(k)}_{\mu/\nu}(x;u) = u^{(k-1)|\mu/\nu|/k} G^{(k)}_{\mu'/\nu'}(x;u^{-1}).$$

6.13. *Proof of Proposition 6.8.* Let $Y_i = Y^{\epsilon_i}$, where ϵ_i is a unit vector, and denote $e_r(Y^{-1}) = e_r(Y_1^{-1},\dots,Y_n^{-1}) = \sum_{|I|=r} Y^{-\epsilon_I}$, where we define $\epsilon_I = \sum_{i\in I}\epsilon_i$ for any subset $I\subseteq\{1,\dots,n\}$. By (17), we have

$$e_r(Y^{-1})\,|\nu+\rho_n\rangle = \sum_{|I|=r} |\nu+\rho+k\epsilon_I\rangle.$$

The vector $\tilde\nu = \nu+\rho_n+k\epsilon_I$ satisfies $\tilde\nu_j < \tilde\nu_i + k$ for all $i<j$. It follows that $|\tilde\nu\rangle$ can be straightened using only those relations in (14) for which $p=0$. As in [16], a simple combinatorial argument then yields

$$\sum_{|I|=r} |\nu+\rho_n+k\epsilon_I\rangle = \sum_{\mu} u^{-(k-1)r+\operatorname{spin}(\mu'/\nu')}\,|\mu+\rho_n\rangle,$$

where the last sum ranges over μ such that μ/ν is a horizontal k-ribbon strip of size kr. The coefficient of any monomial x^λ in $G^{(k)}_{\mu'/\nu'}(x;u)$ is therefore given by

$$u^{-(k-1)|\mu/\nu|/k}\,\langle x^\lambda\rangle G^{(k)}_{\mu'/\nu'}(x;u) = \big\langle|\mu+\rho_n\rangle\big\rangle e_\lambda(Y^{-1})\,|\nu+\rho_n\rangle,$$

where $e_\lambda = e_{\lambda_1}\cdots e_{\lambda_m}$ (this is the proof in [16] that $G^{(k)}_{\mu'/\nu'}(x;u)$ is symmetric). Denote the monomial symmetric functions by m_λ. Using Lemma 6.12, the Cauchy identity $\sum_\lambda m_\lambda(x)e_\lambda(y) = \sum_\lambda s_{\lambda'}(x)s_\lambda(y)$ and the identification $\chi^*_\lambda = s_\lambda(Y^{-1})$, we obtain

$$\begin{aligned} G^{(k)}_{\mu/\nu}(x;u^{-1}) = u^{-(k-1)|\mu/\nu|/k}\omega G^{(k)}_{\mu'/\nu'}(x;u) &= \omega\sum_\lambda m_\lambda(x)\,\big\langle|\mu+\rho_n\rangle\big\rangle e_\lambda(Y^{-1})\,|\nu+\rho_n\rangle \\ &= \sum_\lambda s_\lambda(x)\,\big\langle|\mu+\rho_n\rangle\big\rangle \chi^*_\lambda\,|\nu+\rho_n\rangle. \end{aligned}$$

Equation (19) then follows using Corollary 6.4(ii). ☐

6.14. We now recall an alternative combinatorial description of LLT polynomials from [7], which we shall see leads to a more natural way to formulate Proposition 6.8.

We identify any skew Young diagram with a subset of $\mathbb{Z}\times\mathbb{Z}$, and define the *content* of the box (i,j) in row j, column i to be $c((i,j)) = i-j$. A *skew shape with contents* is an equivalence class of skew Young diagrams up to content-preserving translations. Let $\boldsymbol{\beta} = (\beta^{(1)},\dots,\beta^{(k)})$ be a tuple of skew shapes with contents. Let

$$\mathrm{SSYT}(\boldsymbol{\beta}) = \mathrm{SSYT}(\beta^{(1)})\times\cdots\times\mathrm{SSYT}(\beta^{(k)}).$$

be the set of tuples $T=(T_1,\dots,T_k)$, where T_i is a semistandard Young tableau of shape $\beta^{(i)}$. We call T a semistandard Young tableau on the tuple of shapes $\boldsymbol{\beta}$. Identifying $\boldsymbol{\beta}$ with the disjoint union $\bigsqcup_i \beta^{(i)}$, we regard T as a function $T\colon \boldsymbol{\beta}\to\mathbb{N}$. Its *inversion number* $\operatorname{inv}(T)$

is the number of pairs of boxes $x \in \beta^{(i)}$, $y \in \beta^{(j)}$ such that $T(x) > T(y)$ and either: $i < j$ and $c(x) = c(y)$, or $j < i$ and $c(y) = c(x) + 1$. As usual, the *weight* of a tableau T is

$$x^T = \prod_{z \in \boldsymbol{\beta}} x_{T(z)}.$$

Definition 6.15. *New variant combinatorial LLT polynomials* are the generating functions

$$\mathcal{G}_{\boldsymbol{\beta}}(x; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\beta})} q^{\mathrm{inv}(T)} x^T.$$

6.16. In order to relate the new variant LLT polynomials to the ones in Definition 6.6, we must first recall something about k-cores and k-quotients. For additional information, see [10], [21], [26].

A partition or its diagram μ is a *k-core* if there is no diagram $\nu \subseteq \mu$ such that μ/ν is a k-ribbon. Starting with any partition μ and then removing as many k-ribbons in succession as possible leaves a k-core $\mathrm{core}_k(\mu)$, which depends only on μ and not on the order in which ribbons are removed. Define the *content* of a k-ribbon to be the content of its lower-right box (the box of maximal content). Given any k-core ν, there are exactly k partitions μ such that μ/ν is a k-ribbon, and for each $i = 1, \ldots, k$, there is one with content $c_i \equiv i - 1 \pmod{k}$.

If $\mathrm{core}_k(\mu) = \nu$, there is a unique k-tuple $\boldsymbol{\beta} = \mathrm{quot}_k(\mu)$ of skew shapes with contents, called the k-quotient of μ, such that (i) each $\beta^{(i)}$ is a partition diagram translated so that the box at the origin has content $(c_i - i + 1)/k$, and (ii) the multiset of integers $kc(x) + i - 1$ for $i = 1, \ldots, k$ and $x \in \beta^{(i)}$ is equal to the multiset of contents of the ribbons in any k-ribbon tiling of μ/ν.[2]

If μ/ν can be tiled by k-ribbons, then $\mathrm{core}_k(\mu) = \mathrm{core}_k(\nu)$ and $\mathrm{quot}_k(\nu) \subseteq \mathrm{quot}_k(\mu)$, allowing us to define $\mathrm{quot}_k(\mu/\nu) = (\beta^{(1)}/\gamma^{(1)}, \ldots, \beta^{(k)}/\gamma^{(k)})$, where $\boldsymbol{\beta} = \mathrm{quot}_k(\mu)$, $\boldsymbol{\gamma} = \mathrm{quot}_k(\nu)$. As a tuple of skew shapes with contents, $\mathrm{quot}_k(\mu/\nu)$ depends only on the skew shape with contents μ/ν, not on the specific partitions μ and ν. The Stanton-White correspondence [26] gives a weight-preserving bijection $Q\colon \mathrm{SSRT}_k(\mu/\nu) \to \mathrm{SSYT}(\mathrm{quot}_k(\mu/\nu))$. In [7] it was shown that there is a constant e depending only on μ/ν such that $\mathrm{spin}(S) = -2\,\mathrm{inv}(Q(S)) + e$ for all $S \in \mathrm{SSRT}_k(\mu/\nu)$, yielding the following result.

Proposition 6.17. *Suppose μ/ν can be tiled by k-ribbons, and let $\mathrm{quot}_k(\mu/\nu) = \boldsymbol{\beta}/\boldsymbol{\gamma}$. Then there is an integer e such that (with $q = u^2$)*

$$\mathcal{G}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(x; q) = u^e G^{(k)}_{\mu/\nu}(x; u^{-1})$$

In particular, $\mathcal{G}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(x; q)$ is a symmetric function, and by Corollary 6.9, the coefficients in its Schur polynomial expansion are polynomials in q with non-negative coefficients.

The next proposition is [21, I, 1, Ex. 8(b)], adapted to our notational conventions.

Proposition 6.18. *Define $\rho_r = (0, -1, \ldots, 1 - r)$. Let μ be a partition with at most n parts, set $\nu = \mathrm{core}_k(\mu)$, and let $c_1, \ldots, c_k$ be the contents associated to ν as in §6.16. Let r_i be*

[2] Some authors define k-quotients using different conventions.

the number of entries of $\mu+\rho_n$ that are congruent to i $(\mathrm{mod}\ k)$, and let $\eta=(\eta_1,\dots,\eta_n)$ be the weakly increasing vector with r_i entries equal to $i-k$, for each $i=1,\dots,k$. With these notations, we have:

(i) The numbers r_i and the vector η depend only on $\mathrm{core}_k(\mu)$.

(ii) Let $\boldsymbol{\beta}=\mathrm{quot}_k(\mu)$. Then $\beta^{(i)}$ is a translate of the diagram of a partition $\alpha^{(i)}$ with at most r_i parts.

(iii) Let $a_i=(c_i-i+1)/k$, so $\beta^{(i)}=((a_i^{r_i})+\alpha^{(i)})/(a_i^{r_i})$. The concatenated vector $\widetilde{\beta}=(\alpha^{(1)}+(a_1^{r_1})+\rho_{r_1})\,|\,\cdots\,|\,(\alpha^{(k)}+(a_k^{r_k})+\rho_{r_k})$ has the property that $\mu+\rho_n$ is a permutation of $\eta+k\widetilde{\beta}$.

Using Propositions 6.17 and 6.18, we can reformulate Proposition 6.8 as follows.

Corollary 6.19. *Let $G=GL_n$, $W_f=S_n$, $W_J=S_{r_1}\times\cdots\times S_{r_k}$, so $L=GL_{r_1}\times\cdots\times GL_{r_k}$. Define $\rho_L=\rho_{r_1}|\cdots|\rho_{r_k}$, where $\rho_r=(0,-1,\dots,1-r)$. Given $\beta,\gamma\in X_+(L)$, write $\beta=\beta^{(1)}|\cdots|\beta^{(k)}$, $\gamma=\gamma^{(1)}|\cdots|\gamma^{(k)}$, where $\beta^{(i)},\gamma^{(i)}$ are weakly decreasing integer vectors of length r_i. Then there is an integer m such that*

$$\mathcal{G}_{(\beta^{(1)}/\gamma^{(1)},\dots,\beta^{(k)}/\gamma^{(k)})}(x;q)=q^m\mathcal{L}^{GL_n}_{L,\beta+\rho_L,\gamma+\rho_L}(q)_{\mathrm{pol}}.$$

Remark 6.20. Proposition 6.8 and Corollary 6.19 express combinatorial LLT polynomials in terms of the formal q-characters $\mathcal{L}^{GL_n}_{L,\beta,\gamma}(q)$. This is also reversible. The coefficient of χ_λ in $\mathcal{L}^{GL_n}_{L,\beta,\gamma}(q)$ is non-zero only when $|\lambda|=|\beta|-|\gamma|$, and it is clear from the structure of the extended affine Hecke algebra that these coefficients coincide with those of $\mathcal{L}^{SL_n}_{L',\beta,\gamma}(q)$, for the Levi subgroup L' of SL_n with the same Weyl group W_J as L. Since the weight lattice of SL_n is the quotient of the weight lattice $\mathbb{Z}^n$ of GL_n by the subgroup of constant vectors (r^n), it follows that upon identifying χ_λ with $s_\lambda(x_1,\dots,x_n)$, we get an identity of LLT q-characters for every integer s:

$$\mathcal{L}^{GL_n}_{L,\beta,\gamma}(q)=(x_1\cdots x_n)^{-s}\mathcal{L}^{GL_n}_{L,\beta+(s^n),\gamma}(q).$$

Using Corollary 6.19, this yields a formula

$$\mathcal{L}^{GL_n}_{L,\beta,\gamma}(q)=\lim_{s\to\infty}\left((x_1\cdots x_n)^{-s}q^{m(s)}\mathcal{G}_{((\beta^{(1)}+(s^{r_1}))/\gamma^{(1)},\dots,(\beta^{(k)}+(s^{r_k}))/\gamma^{(k)})}(x_1,\dots,x_n;q)\right).$$

The factors $q^{m(s)}$ are determined, up to an overall common factor q^m, by the requirement that the coefficient of each Schur polynomial $s_\lambda(x_1,\dots,x_n)$ in the argument of the limit is eventually constant for large s.

7. Relation between LLT and generalized Hall-Littlewood polynomials

In this section we derive a formula (Theorem 7.6) expressing the LLT polynomial $\mathcal{L}^G_{L,\beta,\gamma}(q)$, for any G and L, as a finite linear combination of q-induced characters $\mathrm{Ind}^G_{L,q^{-1}}(\chi_\mu(L))$ known as *generalized Hall-Littlewood polynomials* (when $L=T$, the latter are dual forms of the ordinary Hall-Littlewood polynomials associated to G). Our formula is well suited to explicit calculation, and shows in particular that $\mathcal{L}^G_{L,\beta,\gamma}(q)$ belongs to the space spanned by dual Hall-Littlewood polynomials.

We also give a criterion which in certain cases implies that an LLT polynomial $\mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(q)$ is equal to a generalized Hall-Littlewood polynomial $\mathrm{Ind}^G_{L,q^{-1}}(\chi_\beta(L) \otimes \chi_\gamma(L)^*)$. For $G = GL_n$, we use this criterion to prove the conjecture of Shimozono and Weyman mentioned in the introduction.

The methods of this section are purely algebraic and independent of the contents of §§2–4. We make no use of our positivity theorems or the geometric realization of the affine Hecke algebra $\mathcal{H}$.

7.1. Fix G and L as in §5, with Weyl groups $W_f \supseteq W_J$ and weight lattice X. We use the multiplicative notation y^λ for a weight $\lambda \in X$ regarded as an element of the group algebra $\mathbb{Z}[q^{\pm 1}]X$. Let $R_+(L)$ be the set of positive roots of L, and define

$$\Delta^L_q = \prod_{\alpha \in R_+(L)} (1 - q y^{-\alpha}), \qquad \Delta^L = \Delta^L_1, \qquad \Delta_q = \Delta^G_q, \qquad \Delta = \Delta_1.$$

Let $*$ denote the involutory ring automorphism of $\mathbb{Z}[q^{\pm 1}]X$ such that

$$q^* = q^{-1}, \qquad (y^\lambda)^* = y^{-\lambda}.$$

We assume that X contains a weight ρ_L such that

$$\langle \alpha_i^\vee, \rho_L \rangle = 1 \quad \text{for all } i \in J. \tag{20}$$

The choice of ρ_L is unique modulo W_J-invariant weights. We will mostly refer to ρ_L in expressions which are independent of this choice, mentioning any exceptions explicitly.

Definition 7.2. *Generalized Hall-Littlewood polynomials* are the formal q-characters of G

$$\mathrm{Ind}^G_{L,q^{-1}}(\chi_\mu(L)) = \sum_{w \in W_f} w\left(\frac{y^\mu\,(\Delta^L_q)^*}{\Delta\,\Delta^*_q}\right), \tag{21}$$

where $\mu \in X_+(L)$. We extend $\mathrm{Ind}^G_{L,q^{-1}}(\chi)$ linearly to all (virtual) characters χ of L.

Remarks 7.3. (a) It is conventional to extend the Weyl character formula and define

$$\chi_\mu(L) = \sum_{w \in W_J} w(y^\mu/\Delta^L) \tag{22}$$

for all $\mu \in X$. Then $\chi_\mu(L) = 0$ if $\mu + \rho_L$ is not regular for L, otherwise $\chi_\mu(L) = (-1)^w \chi_{w(\mu+\rho_L)-\rho_L}(L)$, where $w \in W_J$ and $w(\mu + \rho_L) \in X_{++}(L)$.

Since $\Delta^L(\Delta^L_q)^*/(\Delta\,\Delta^*_q)$ is W_J-invariant, the right-hand side of (21) is a function of the expression in (22). In other words, the identity

$$\mathrm{Ind}^G_{L,q^{-1}}\Big(\sum_{w \in W_J} w(f/\Delta^L)\Big) = \sum_{w \in W_f} w\left(\frac{f\,(\Delta^L_q)^*}{\Delta\,\Delta^*_q}\right)$$

holds for $f = y^\mu$, hence for all f. If g is W_J-invariant, it follows that

$$\operatorname{Ind}_{L,q^{-1}}^G(\chi_\mu(L)\, g) = \sum_{w\in W_f} w\left(\frac{y^\mu\, g\,(\Delta_q^L)^*}{\Delta\,\Delta_q^*}\right). \tag{23}$$

(b) It is immediate that $\operatorname{Ind}_{L,q^{-1}}^G(\chi_\mu(L))$ is a formal power series in q^{-1} with finite G-characters as coefficients.

(c) Define the generalized q-Kostant partition function by

$$\mathcal{P}_L^G(\lambda; q) \underset{\text{def}}{=} \langle y^{-\lambda}\rangle \left(\Delta_q^L/\Delta_q\right).$$

Then $\mathcal{P}_L^G(\lambda; q)$ is the generating function enumerating multisets M of roots in $R_+(G)\setminus R_+(L)$ such that $\lambda = \sum M$, with weight $q^{|M|}$. By the Weyl character formula, the coefficient of χ_λ in $\operatorname{Ind}_{L,q^{-1}}^G(\chi_\mu(L))$ for any $\lambda \in X_+$ is given by

$$\langle \chi_\lambda\rangle \operatorname{Ind}_{L,q^{-1}}^G(\chi_\mu(L)) = \sum_{w\in W_f} (-1)^{l(w)} \mathcal{P}_L^G(w(\lambda+\rho_G) - \rho_G - \mu; q^{-1}). \tag{24}$$

In particular, since the sum is finite, $\operatorname{Ind}_{L,q^{-1}}^G(\chi_\mu(L))$ is a formal q-character, as asserted.

(d) The expression in (24) specializes at $q = 1$ to the multiplicty of $\chi_\mu(L)$ in the restriction $\chi_\lambda(G)|L$. It is therefore natural to regard $\operatorname{Ind}_{L,q^{-1}}^G(\chi_\mu(L))$ as a q-induced character from L to G, as our notation suggests.

(e) When $L = T$ and $\mu \in X_+$, the expression in (24) is Lusztig's *q-analog of weight multiplicity* $K_{\lambda\mu}(q^{-1})$. By Kato [12], the $K_{\lambda\mu}(q)$ are the coefficients in the expansion of χ_λ through the usual Hall-Littlewood polynomials $P_\mu(y; q)$ for G. In other words, the formal q-characters $\operatorname{Ind}_{T,q}^G(y^\mu)$ for $\mu \in X_+$ are the dual basis to the polynomials $P_\mu(y; q)$, with respect to the standard inner product such that $\langle \chi_\lambda, \chi_\nu\rangle = \delta_{\lambda\nu}$.

(f) Using the fact that $(1 - qy^{-\alpha_i})/\Delta_q$ is s_i-invariant, one easily verifies the identity

$$\operatorname{Ind}_{T,q^{-1}}^G(y^\mu - q^{-1}y^{\mu+\alpha_i} + y^{s_i(\mu)-\alpha_i} - q^{-1}y^{s_i(\mu)}) = 0.$$

If $\langle \alpha_i^\vee, \mu\rangle < 0$, this expresses $\operatorname{Ind}_{T,q^{-1}}^G(y^\mu)$ as a linear combination of other terms $\operatorname{Ind}_{T,q^{-1}}^G(y^\nu)$ such that $\nu < \mu$ in the Bruhat order on $X = W/W_f$. For every $\mu \in X$, it follows that $\operatorname{Ind}_{T,q^{-1}}^G(y^\mu)$ lies in the subspace of all formal q-characters of G spanned by the dual Hall-Littlewood polynomials $\operatorname{Ind}_{T,q^{-1}}^G(y^\nu)$ for $\nu \in X_+$. Expanding the factor $(\Delta_q^L)^*$ in the numerator of (21), we see that $\operatorname{Ind}_{L,q^{-1}}^G(\chi_\mu(L))$ also lies in this subspace, for every L and every $\mu \in X$.

(g) Let $P \subseteq G$ be a parabolic subgroup with Levi factor L, $V(\mu)$ an irreducible L-module regarded as a P-module, and V_μ the vector bundle $G \times_P V(\mu)$ on G/P. Let $T = T(G/P)$ be the tangent bundle. The group G acts equivariantly on V_μ and T, while $\mathbb{C}^*$ acts on T by homotheties. Identifying the characters of $\mathbb{C}^*$ with powers of q, we have

$$\operatorname{Ind}_{L,q^{-1}}^G(\chi_\mu(L))^* = \sum_i (-1)^i \operatorname{ch} H^i(G/P, V_\mu^* \otimes \operatorname{Sym}(T)), \tag{25}$$

where ch denotes the character of a $(G \times \mathbb{C}^*)$-module.

(h) When $L = T$ and $\mu \in X_+$, Hesselink [9] showed that the higher cohomology groups in (25) vanish, as can also be seen using the Grauert–Riemenschneider vanishing theorem [5]. In particular, the coefficients $K_{\lambda\mu}(q^{-1}) = \langle \chi_\lambda \rangle \operatorname{Ind}_{T,q^{-1}}^G(y^\mu)$ for $\lambda, \mu \in X_+$ are positive. This positivity also follows from the theorem of Kato [12] that $K_{\lambda\mu}(q^{-1})$ is a power of q times a Kazhdan-Lusztig polynomial for the affine Weyl group W. Our results below generalize Kato's theorem—see §7.14.

Broer (see [24, Conjecture 5]) conjectured that for $\mu \in X_+$, the higher cohomology groups in (25) vanish for any L, and consequently that the *generalized Kostka coefficients* $K^G_{L,\lambda,\mu}(q^{-1}) = \langle \chi_\lambda \rangle \operatorname{Ind}_{L,q^{-1}}^G(\chi_\mu(L))$ are positive. For $G = GL_n$, Shimozono and Weyman [24] conjectured an explicit combinatorial rule for these coefficients. These conjectures remain open.

7.4. Although everything we will define depends only on $q = u^2$, we now take coefficients in $\mathbb{Z}[u^{\pm 1}]$ for compatibility with §5, and keep the notation $\mathcal{H}$, Y^λ, $\mathcal{Y}$, e^+, e_J^- introduced there. We identify $\mathbb{Z}[u^{\pm 1}]X$ with the left $\mathcal{H}$-module $\mathcal{Y}e^+$, setting $y^\lambda = Y^\lambda e^+$. Formula (9) implies that the $\mathcal{H}$ action is given explicitly by the Demazure–Lusztig operators

$$T_{s_i} = q s_i + (q-1)\frac{1}{1 - y^{-\alpha_i}}(1 - s_i) \qquad (i \neq 0). \tag{26}$$

For any $\lambda \in X$, let $\lambda = w(\lambda_+)$, where $\lambda_+ \in X_+$ and $w \in W_f$. Also, let $\lambda = v(\lambda_-)$, where $\lambda_- \in -X_+$ and v is the *minimal* element of W_f with this property. Then define

$$\begin{aligned} E_\lambda &= q^{-l(w)} T_w(y^{\lambda_+}) \\ F_\lambda &= T_v(y^{\lambda_-}). \end{aligned} \tag{27}$$

These formulae are normalized to make E_λ and F_λ "monic," *i.e.*, $\langle y^\lambda \rangle E_\lambda = \langle y^\lambda \rangle F_\lambda = 1$. Since $q^{-1}T_{s_i}$ fixes y^μ if $s_i\mu = \mu$, the formula for E_λ does not depend on the choice of w, but the choice of v does affect the normalization of F_λ.

7.5. Fixing G and L, define

$$(f \star g) = \operatorname{Ind}_{L,q^{-1}}^G\left(\frac{\sum_{v,w \in W_J}(-1)^{l(vw)} v(f) w(g^*)}{\Delta^L (\Delta^L)^*}\right). \tag{28}$$

With the convention of Remark 7.3(a) about characters indexed by non-dominant weights, $(- \star -)$ can be characterized as the unique skew-bilinear function from $\mathbb{Z}[u^{\pm 1}]X \times \mathbb{Z}[u^{\pm 1}]X$ to formal G-characters such that $(y^\beta \star y^\gamma) = \operatorname{Ind}_{L,q^{-1}}^G(\chi_{\beta-\rho_L}(L) \otimes \chi_{\gamma-\rho_L}(L)^*)$ for all $\beta, \gamma \in X$.

Theorem 7.6. *With the above notation, there is an integer m (depending on β, γ) such that*

$$\mathcal{L}^G_{L,\beta,\gamma}(q) = q^m (E_\beta \star F_\gamma). \tag{29}$$

Corollary 7.7. *The formal q-characters $\mathcal{L}^G_{L,\beta,\gamma}(q)$ lie in the subspace spanned by the dual Hall-Littlewood polynomials $\operatorname{Ind}_{T,q^{-1}}^G(y^\mu)$, $\mu \in X_+$. In particular, $\mathcal{L}^G_{L,\beta,\gamma}(q)$ is a Laurent series in q^{-1} with finite G-characters as coefficients.*

Remark 7.8. It follows from the definitions that $F_\gamma + F_{s_i(\gamma)}$ and $E_\beta + qE_{s_i(\beta)}$ for $\langle \alpha_i^\vee, \beta\rangle \geq 0$ are s_i-invariant. Since $(- \star -)$ is W_J-antisymmetric in each variable, we therefore have

$$(E_\beta \star F_\gamma) = (-1)^{l(v)}(-q)^{l(u)}(E_{u(\beta)} \star F_{v(\gamma)}) \tag{30}$$

for $\beta \in X_+(L)$, $\gamma \in X$, and $u, v \in W_J$. In particular, (29) is equivalent to

$$\mathcal{L}^G_{L,\beta,\gamma}(q) = (-1)^{l(w_0^J)} q^m (E_\beta \star F_{w_0^J(\gamma)}). \tag{31}$$

The latter is useful for computations because $F_{w_0^J}(\gamma)$ typically has fewer terms than F_γ.

7.9. *Proof of Theorem 7.6.* The proof will be in three steps. First we interpret the coefficients of $\mathcal{L}^G_{L,\beta,\gamma}(q)$ as matrix coefficients of χ_λ on the elements $\{e_J^- F_\gamma\}$. Next we show that $\{E_\lambda\}$ and $\{F_\lambda\}$ are dual bases for a skew-bilinear inner product $(-,-)$ on $\mathbb{Z}[u^{\pm 1}]X$ given by an explicit formula. Finally we give a formula for the operator e_J^- acting on $\mathbb{Z}[u^{\pm1}]X$. Combining these formulas will yield the formula asserted by the theorem.

To avoid bothering with stray powers of u and q in what follows, we write $f \sim g$ to mean that $f = u^m g$ for some integer m. Under the identification $y^\lambda = Y^\lambda e^+$, we have

$$E_\lambda \sim T_w e^+ \quad \text{for } w \in \tau(\lambda) W_f \tag{32}$$

$$\overline{E_\lambda} \sim F_\lambda. \tag{33}$$

The first identity follows directly from the definitions, using the fact that for $\lambda_+ \in X_+$ and $w \in W_f$, the product $w\tau(\lambda_+)$ is reduced. For the second, let $\lambda = v(\lambda_-) = vw_0(\lambda_+)$, where we assume that v is minimal. Then $E_\lambda \sim T_{vw_0} T_{\tau(\lambda_+)} e^+ = \overline{T_v} T_{w_0\tau(\lambda_+)} e^+ \sim \overline{T_v} T_{\tau(\lambda_-)} e^+$, and (33) follows, since $\overline{e^+} = e^+$.

By (32), the coefficient $\langle \chi_\lambda \rangle \mathcal{L}^G_{L,\beta,\gamma}(q)$ is equal to the coefficient of $e_J^- E_\beta$ in $\chi_\lambda e_J^- E_\gamma$, times a power of q independent of λ. Now, $\overline{\chi_\lambda} = \chi_\lambda$ by Corollary 5.6, and $\overline{e_J^-} = e_J^-$, hence (33) implies

$$\mathcal{L}^G_{L,\beta,\gamma}(q^{-1}) \sim \sum_\lambda \chi_\lambda \cdot \langle e_J^- F_\beta \rangle (\chi_\lambda \, e_J^- F_\gamma).$$

Moreover, since $\beta \in X_{++}(L)$, we have

$$e_J^- F_\beta = \overline{e_J^-} F_\beta = (-u)^{-l(w_0^J)} \sum_{w \in W_J} (-q)^{l(w)} T_{w^{-1}}^{-1} F_\beta = (-u)^{-l(w_0^J)} \sum_{w \in W_J} (-q)^{l(w)} F_{w(\beta)},$$

and therefore

$$\mathcal{L}^G_{L,\beta,\gamma}(q^{-1}) \sim (-1)^{l(w_0^J)} \sum_\lambda \chi_\lambda \cdot \langle F_\beta \rangle (\chi_\lambda \, e_J^- F_\gamma). \tag{34}$$

Next, define a skew-bilinear form $(-,-)^\sim$ on $\mathcal{H}$ by setting

$$(f, g)^\sim = \langle T_1 \rangle (\widetilde{g}\, f),$$

where $\widetilde{\cdot}$ is the $\mathbb{Z}$-algebra anti-automorphism of $\mathcal{H}$ such that $\widetilde{u} = u^{-1}$ and $\widetilde{T}_w = T_w^{-1} = \overline{T_{w^{-1}}}$ for $w \in W$. Define $(-,-)$ to be the restriction of $(1/W_f(q^{-1}))(-,-)^\sim$ to $\mathbb{Z}[u^{\pm 1}]X = \mathcal{Y}e_+ \subseteq \mathcal{H}$, where $W_f(q^{-1}) = \sum_{w\in W_f} q^{-l(w)} = (e^+, e^+)\tilde{}$. We claim that

$$(E_\beta, F_\gamma) = \delta_{\beta\gamma} \quad \text{for all } \beta, \gamma \in X; \tag{35}$$

$$(f, g) = \langle 1\rangle\, (\Delta/\Delta_q)^*\, g^* f \quad \text{for all } f, g \in \mathbb{Z}[u^{\pm 1}]X, \tag{36}$$

where $1/\Delta_q^*$ is understood as a formal power series in the elements y^{α_i} ($i \neq 0$). Note that only a finite number of terms of this series contribute to the value of (f, g).

For (35), recall that $\langle T_1\rangle(T'_v T'_w) = \delta_{v,w^{-1}}$ in $\mathcal{H}$, which implies $(T'_v, \overline{T'_w})\tilde{} = \delta_{vw}$. Then (32) and (33) immediately imply $(E_\beta, F_\gamma) = 0$ if $\beta \neq \gamma$. The value $(E_\beta, F_\beta) = 1$ will follow from (36), since $E_\beta F_\beta^* \equiv 1$ modulo terms involving only y^ν with $\nu \in Q_+ \setminus \{0\}$. For a justification of this last point, see (40)–(41) in §7.11, below.

To prove (36), it suffices to consider the case $f = y^\beta$, $g = y^\gamma$, as both sides are skew-bilinear. We have

$$(y^\beta, y^0) = (E_\beta, F_0) = \delta_{\beta,0} \quad \text{for } \beta \in X_+,$$

since $(1, 1) = 1$ by the normalization we chose for $(-,-)$. The identity $(T_{s_i} f, T_{s_i} g)\tilde{} = (f, g)\tilde{}$ for all i (including $i = 0$) is immediate from the definition. This unitarity property implies

$$(y^\beta, y^\gamma) = (y^{\beta-\gamma}, 1) \quad \text{for all } \beta, \gamma \in X;$$
$$q(y^\lambda, 1) - (y^{\lambda+\alpha_i}, 1) + q(y^{s_i(\lambda)-\alpha_i}, 1) - (y^{s_i(\lambda)}, 1) = 0 \quad \text{for } i \neq 0 \text{ and all } \lambda \in X.$$

To see this, use the unitarity of Y^γ for the first identity, and apply $(T_{s_i} f, 1) = (f, T_{s_i}^{-1} 1) = q(f, 1)$ with $f = y^\lambda(1 - y^{\alpha_i})$ for the second. The three conditions above characterize (y^β, y^γ) uniquely. One checks directly that they also hold for the right-hand side of (36).

The third step in our proof is the following formula for e_J^- acting on $\mathbb{Z}[u^{\pm 1}]X$:

$$e_J^- = (-u)^{-l(w_0^J)} \frac{\Delta_q^L}{\Delta^L} \sum_{w\in W_J} (-1)^{l(w)}\, w. \tag{37}$$

To prove it, first observe that if $s_i f = f$ for some $i \in J$, or equivalently, $T_i f = qf$, then the operator on each side of (37) kills f. This given, it suffices to verify that if $T_i f = -f$ for all $i \in J$, then the operator A on the right-hand side of (37) satisfies $Af = e_J^- f = (-u)^{-l(w_0^J)} W_J(q) f$. One checks that $T_i f = -f$ if and only if $h = ((1 - y^{-\alpha_i})/(1 - qy^{-\alpha_i}))\, f$ satisfies $s_i h = -h$. This holds for all $i \in J$ if and only if $g = (\Delta^L/\Delta_q^L) f$ satisfies $s_i g = -g$ for all $i \in J$. Then

$$\sum_{w\in W_J} (-1)^{l(w)} w(f) = \left(\sum_{w\in W_J} w(\Delta_q^L/\Delta^L)\right) g = W_J(q)\, g$$

by Macdonald's identity [20, (2.8)], which yields the desired value for Af.

Now we combine the preceding conclusions. First, (34) and (35) imply

$$L^G_{L,\beta,\gamma}(q) \sim (-1)^{l(w_0^J)} \sum_\lambda \chi_\lambda \cdot (E_\beta, \chi_\lambda e_J^- F_\gamma).$$

Note that q^{-1} in (34) has been replaced by q because the functional $(E_\beta, -)$ is skew-linear. The desired formula (29) follows once we establish the identity

$$\langle \chi_\lambda \rangle \operatorname{Ind}_{L,q^{-1}}^G(\chi_{\beta-\rho_L}(L) \otimes \chi_{\gamma-\rho_L}(L)^*) \sim (-1)^{l(w_0^J)}(y^\beta, \chi_\lambda\, e_J^- y^\gamma)$$

with constant of proportionality independent of λ. To this end, note that the identity

$$\langle \chi_\lambda \rangle \sum_{w \in W_f} w(f/\Delta) = \langle 1 \rangle (\chi_\lambda \Delta)^* f$$

holds for all f, as one sees by taking $f = y^\mu$. Hence for any W_f-invariant g, we have

$$\langle \chi_\lambda \rangle \operatorname{Ind}_{L,q^{-1}}^G(\chi_\mu(L)\, g) = \langle 1 \rangle (\Delta \Delta_q^L / \Delta_q)^* \chi_\lambda^*\, y^\mu g$$

in consequence of (23). Taking $\mu = \beta - \rho_L$ and $g = \chi_{\gamma-\rho_L}(L)^*$, this gives

$$\begin{aligned}\langle \chi_\lambda \rangle \operatorname{Ind}_{L,q^{-1}}^G(\chi_{\beta-\rho_L}(L) \otimes \chi_{\gamma-\rho_L}(L)^*) &= \langle 1 \rangle \left(\frac{\Delta\, \Delta_q^L}{\Delta_q\, \Delta^L} \right)^* y^\beta \chi_\lambda^* \sum_{w \in W_J} (-1)^{l(w)} y^{-w(\gamma)} \\ &= (-u)^{-l(w_0^J)} (y^\beta, \chi_\lambda\, e_J^- y^\gamma)\end{aligned}$$

by (36) and (37), completing the proof. □

Remark 7.10. We should point out that E_λ and F_λ are *non-symmetric Hall-Littlewood polynomials*, and that the argument in §7.9 is in essence a non-symmetric version of the methods of Kato [12]. The E_λ and F_λ are also specializations of the non-symmetric Macdonald polynomials $E_\lambda(y; q, t)$ as $q \to 0$ or ∞, with t replaced by q^{-1}, which gives another way to see why (35) holds for the inner product defined by (36).

7.11. We conclude with a sufficient condition for Theorem 7.6 to yield an identity

$$\mathcal{L}_{L,\beta,\gamma}^G(q) = q^m \operatorname{Ind}_{L,q^{-1}}^G(\chi_{\beta-\rho_L}(L) \otimes \chi_{\gamma-\rho_L}(L)^*) \tag{38}$$

between an LLT polynomial and a single generalized Hall-Littlewood polynomial. Let a_L denote the antisymmetrization operator

$$a_L f = \sum_{w \in W_J} (-1)^{l(w)} w(f).$$

Since $(f \star g)$ is a function of $a_L f$ and $a_L g$, it is immediate from (31) that (38) will hold if

$$a_L E_\beta = a_L y^\beta \quad \text{and} \quad a_L F_{w_0^J(\gamma)} = a_L y^{w_0^J(\gamma)}. \tag{39}$$

Our aim is to articulate a more combinatorial criterion that implies (39).

Define the *support* of an element $f \in \mathbb{Z}[u^{\pm 1}]X$ to be the set of weights with non-zero coefficient in f,

$$\operatorname{supp}(f) = \{\mu \in X : \langle y^\mu \rangle f \neq 0\}.$$

Given a subset $U \subseteq X$, all of whose elements belong to a single coset $\lambda + Q$ of the root lattice, let $\operatorname{conv}(U)$ denote the convex hull of U in $\lambda + Q$. In particular, for any $\lambda \in X_+$, we have $\operatorname{supp}(\chi_\lambda) = \operatorname{conv}(W\lambda)$. From (26) we see that $\operatorname{supp}(T_{s_i}y^\lambda) \subseteq \operatorname{conv}(\lambda, s_i(\lambda))$ for all λ, which implies

$$\operatorname{supp}(T_w y^\lambda) \subseteq \operatorname{conv}(\{v(\lambda) : v \leq w\}) \tag{40}$$

by induction on $l(w)$. Note also that if $\lambda \in X_+$, we have

$$\operatorname{conv}(\{v(\lambda) : v \leq w\}) \subseteq (w(\lambda) + Q_+) \cap \operatorname{conv}(W\lambda), \tag{41}$$

where $Q_+ \subseteq Q$ is the submonoid generated by the positive roots, since the set on the right is convex and contains all the elements $v(\lambda)$ on the left.

Definition 7.12. A regular dominant weight $\beta \in X_{++}(L)$ for L is *L-quasi-dominant* if

$$X_{++}(L) \cap (\beta + Q_+) \cap \operatorname{conv}(W\beta) = \{\beta\}$$

(in particular, this holds if β is dominant for G).

Note that if $(\beta + Q_+) \cap \operatorname{conv}(W\beta)$ contains a weight μ, then it also contains the unique weight $\mu_L^+ \in X_+(L) \cap W_J(\mu)$. If β is L-quasi-dominant, it therefore follows that β is the only L-regular weight in $(\beta + Q_+) \cap \operatorname{conv}(W\beta)$. Together with (40)–(41), this implies that $a_L E_\beta = a_L y^\beta$. Similarly, if $-\gamma$ is L-quasi-dominant, then $a_L F_\gamma = a_L y^\gamma$, yielding the following result.

Proposition 7.13. *Let $\beta, \gamma \in X_{++}(L)$ be such that β and $-w_0^J(\gamma)$ are both L-quasi-dominant. Then there is an integer m such that*

$$\mathcal{L}^G_{L,\beta,\gamma}(q) = q^m \operatorname{Ind}^G_{L,q^{-1}}(\chi_{\beta-\rho_L}(L) \otimes \chi_{\gamma-\rho_L}(L)^*).$$

Remark 7.14. When $L = T$, we can take $\gamma = \rho_L = 0$ and $\beta \in X_+$ to see that the ordinary dual Hall-Littlewood polynomial $\operatorname{Ind}^G_{T,q^{-1}}(y^\beta)$ is a power of q times the LLT polynomial $\mathcal{L}^G_{T,\beta,0}(q)$, a result equivalent to Kato's theorem mentioned in Remark 7.3(h). Our proof in this case specializes to the one given by Kato.

We now apply the criterion in Proposition 7.13 to prove the following conjecture of Shimozono and Weyman [24, §1, p. 258] equating certain generalized Hall-Littlewood polynomials for GL_n with LLT polynomials.

Theorem 7.15. *Let $G = GL_n$, $L = GL_{r_1} \times \cdots \times GL_{r_k}$. Let $\mu \in X_+$ be W_J-invariant, so $\mu = (m_1^{r_1}, \ldots, m_k^{r_k})$, where $m_1 \geq \cdots \geq m_k$, and assume $m_k \geq 0$. Let $\boldsymbol{\beta}/\boldsymbol{\gamma}$ be a k-tuple of rectangular Young diagrams with contents such that $\beta^{(i)}/\gamma^{(i)}$ is a translate of $(m_i^{r_i})$, the contents a_i of the southwest corners of the rectangles $\beta^{(i)}/\gamma^{(i)}$ are weakly increasing, and the contents $a_i + m_i - 1$ of their southeast corners are weakly decreasing. Then there is an integer m such that*

$$\operatorname{Ind}^{GL_n}_{L,q^{-1}}(\chi_\mu(L))_{\mathrm{pol}} = q^m \mathcal{G}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(x; q). \tag{42}$$

In fact, we will prove something stronger. Fix $\rho_L = \rho_{r_1} | \cdots | \rho_{r_k}$, where we set $\rho_r = (0, -1, \ldots, 1-r)$, as in Corollary 6.19. Let $\beta^{(i)} = (a_i^{r_i}) + (m_i^{r_i})$, $\gamma^{(i)} = (a_i^{r_i})$, where m_i, a_i

are as in the statement of the theorem, and put $\beta = \beta^{(1)}|\cdots|\beta^{(k)}$, $\gamma = \gamma^{(1)}|\cdots|\gamma^{(k)}$. We will show that even before trunctation to polynomial characters, there is an m such that

$$\operatorname{Ind}_{L,q^{-1}}^{GL_n}(\chi_\mu(L)) = q^m \mathcal{L}^{GL_n}_{L,\beta+\rho_L,\gamma+\rho_L}(q) \tag{43}$$

for this particular choice of ρ_L. Then Corollary 6.19 gives (42). Note that $\chi_\beta(L)$, $\chi_\gamma(L)$ and $\chi_\mu(L)$ are one-dimensional, and $\chi_\mu(L) = \chi_\beta(L) \otimes \chi_\gamma(L)^*$. Thus (43) is an immediate consequence of Proposition 7.13 and the following lemma.

Lemma 7.16. *Let $G = GL_n$, let L and $\rho_L = \rho_{r_1}|\cdots|\rho_{r_k}$ be as above, and let $\beta = (b_1^{r_1})|\cdots|(b_k^{r_k})$, $\gamma = (a_1^{r_1})|\cdots|(a_k^{r_k})$.*

(i) If $b_1 \geq \cdots \geq b_k$, then $\beta + \rho_L$ is L-quasi-dominant.

(ii) if $a_1 \leq \cdots \leq a_k$ then $-w_0^J(\gamma + \rho_L)$ is L-quasi-dominant.

Proof. A weight $\lambda \in X$ is L-quasi-dominant if and only if $-w_0(\lambda)$ is L_1-quasi-dominant, where $L_1 = w_0(L) = GL_{r_k} \times \cdots \times GL_{r_1}$. Hence (i) with L_1 in place of L implies (ii).

For (i), set $\lambda = \beta + \rho_L$ and suppose that $\mu \in X_{++}(L) \cap (\lambda + Q_+) \cap \operatorname{conv}(W\lambda)$. In particular, $\mu \in \lambda_+ - Q_+$, where λ_+ is the dominant weight in the orbit $W\lambda$. By hypothesis, $\lambda_1 = \max_i(\lambda_i) = (\lambda_1)_+$. Hence $\mu_1 = \lambda_1$, since $\mu \in (\lambda + Q_+) \cap (\lambda_+ - Q_+)$. The condition $\mu \in X_{++}(L)$ then implies $\mu_i \leq \lambda_1 - i + 1 = \lambda_i$ for $i = 1, \ldots, r_1$. Moreover, $\mu \in (\lambda + Q_+)$ implies an opposite inequality, $\mu_1 + \cdots + \mu_j \geq \lambda_1 + \cdots + \lambda_j$, for all j, whence

$$(\mu_1, \ldots, \mu_{r_1}) = (\lambda_1, \ldots, \lambda_{r_1}). \tag{44}$$

Let W' and Q' be the Weyl group and root lattice of GL_{n-r_1}, and set $\underline{\lambda} = (\lambda_{r_1+1}, \ldots, \lambda_n)$ and $\underline{\mu} = (\mu_{r_1+1}, \ldots, \mu_n)$. We claim that (44) implies $\underline{\mu} \in (\underline{\lambda} + Q'_+) \cap \operatorname{conv}(W'\underline{\lambda})$. This given, the desired result $\mu = \lambda$ will follow by induction on k.

For the claim, recall that $\mu \in \lambda + Q_+$ if and only if $|\mu| = |\lambda|$ and $\mu_1 + \cdots + \mu_j \geq \lambda_1 + \cdots + \lambda_j$ for all j. Then $\underline{\mu} \in \underline{\lambda} + Q'_+$ is clear.

After perhaps adding a constant to all components of μ and λ, we can assume that μ_+ and λ_+ are partitions. The condition $\mu \in \operatorname{conv}(W\lambda)$ is equivalent to $\mu_+ \in \lambda_+ - Q_+$, that is, to $\mu_+ \leq \lambda_+$ in the standard "dominance" partial ordering on partitions. It is immediate from the definition that the dominance ordering satisfies $\nu \leq \theta$ if and only if $\nu + (1^m) \leq \theta + (1^m)$. Since transpose reverses dominance—that is, $\nu \leq \theta$ if and only if $\nu' \geq \theta'$ [21, (1.11)]—it follows that for any m we have $\nu \leq \theta$ if and only if $(m, \nu)_+ \leq (m, \theta)_+$. Iterating this, we see that $\mu_+ \leq \lambda_+$ implies $\underline{\mu}_+ \leq \underline{\lambda}_+$. Therefore $\underline{\mu} \in \operatorname{conv}(W'\underline{\lambda})$, as claimed. □

Remark 7.17. Allowing G and L to be arbitrary once again, let $\alpha \in X_+$ be a dominant weight for G. Suppose that there exists a choice of ρ_L satisfying (20) such that both $\alpha + \rho_L$ and $-w_0^J(\rho_L)$ are quasi-dominant for L. In this case, Proposition 7.13 shows that

$$\operatorname{Ind}_{L,q^{-1}}^{G}(\chi_\alpha(L)) = q^m \mathcal{L}^{G}_{L,\alpha+\rho_L,\rho_L}(q), \tag{45}$$

and Theorem 5.9 then implies that the generalized Kostka coefficients $\langle \chi_\lambda \rangle \operatorname{Ind}_{L,q^{-1}}^{G}(\chi_\alpha(L))$ are positive, as predicted by the conjecture of Broer in Remark 7.3(h). However, examples exist in which (45) *does not hold for any choice of ρ_L*.

One such example is $G = GL_7$, $L = GL_3 \times GL_4$, $\alpha = (3,3,2)|(2,2,1,1)$. In this example, even the truncation of (45) to polynomial characters fails for all choices of ρ_L, as one can verify with the help of Theorem 7.6 and a computer. This example is not the simplest one possible, but we have chosen it for its additional interesting property that $\mathrm{Ind}^G_{L,q^{-1}}(\chi_\alpha(L))_{\mathrm{pol}}$ is a *k-split polynomial* (with $k = 5$) in the sense of Lapointe and Morse [15], showing that even these rather special generalized Hall-Littlewood polynomials do not always coincide with LLT polynomials.

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(I.G.) DPMMS, Centre for Mathematical Sciences, University of Cambridge, Cambridge, England

(M.H.) Dept. of Mathematics, University of California, Berkeley, California, US

E-mail address, I.G.: `i.grojnowski@dpmms.cam.ac.uk`

E-mail address, M.H.: `mhaiman@math.berkeley.edu`