CONSTRUCTING THE ASSOCIAHEDRON

Mark Haiman

The following pages contain scanned images of a manuscript written in the fall of 1984. The purpose of the manuscript was to prove that the combinatorial incidence relations between triangulations and chords of an n-gon are realized by the incidence relations between facets and vertices of a polytope (called variously the Stasheff polytope, the Tamari polytope or the associahedron).

I never published the manuscript because it was soon made obsolete, first by a simpler and more symmetric construction of such a polytope by Carl Lee, and subsequently by the more general theory of secondary polytopes (the associahedron is secondary polytope of the n-gon). I am reproducing it here because various people have inquired about the manuscript for historical reasons.

Please keep in mind that the manuscript is only a draft. It gives no references, and contains at least one error that I know of: the remark at the bottom of page 4 should apply only to *regular* decompositions of an arbitrary polytope. From the theory of secondary polytopes, of course, we now know that the answer to the question raised in that remark is "yes."

Constructing the Associahedron

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The associahedron is a mythical polytope whose face structure represents the lattice of partial parenthesizations of a sequence of variables, in a way to be made precise below. The purpose of these notes is to give an explicit construction of such a polytope.

Let $x_1, \ldots, x_n$ be variables. A bracket is a consecutive subsequence of the x_i. Denote the bracket $\{x_i, \ldots, x_j\}$ by $[i, j]$. A grouping G is a set of brackets such that

i) every two are either nested or disjoint.

ii) each is either a singleton or the union of two smaller brackets in G.

iii) $[1, n] \in G$.

$[1, n]$ and singletons are improper brackets; note that every grouping contains all improper brackets.

An associahedron will now be a polytope whose vertices correspond to groupings and whose facets correspond to brackets, in such a way that the incidence relation between groupings and brackets is reflected in its facet-vertex incidence relation.

Let V be a real vector space with $x_1, \dots, x_n$ as a basis. Let $e_1, \dots, e_n$ be the dual basis of V^*. Fix a number $0 < \tau < 1$ and define for each bracket a functional

$$l_{[i,j]} = \sum_{i \le k < j} \tau^{k-i} e_k - e_j$$

<u>A1</u>. Let G be a grouping. Then $\{ l_{[i,j]} \mid [i,j] \in G,\ i \ne j \} \cup \{e_n\}$ is a basis of V^*.

<u>Proof</u>. By induction on n. For $n=1$, this is trivial. For $n > 1$ let $[1,n] = [1,m] \cup [m+1, n]$ (both $\in G$). By induction,

$$\{ l_{[i,j]} \mid [i,j] \in G,\ [i,j] \subseteq [1,m],\ i \ne j \} \cup \{e_m\} \cup$$
$$\{ l_{[i,j]} \mid [i,j] \in G,\ [i,j] \subseteq [m+1,n],\ i \ne j \} \cup \{e_n\}$$

is a basis of V^*.

Now,

$$l_{[1,n]} = l_{[1,m]} + (1+\tau^{m-1})e_m + \tau^m l_{[m+1,n]} - (1-\tau^m)e_n.$$

Since $1+\tau^{m-1} \neq 0$, we may exchange e_m for $l_{[1,n]}$ and still have a basis of V^*. ▨

A2. The simplicial complex whose points are the proper brackets and whose simplices are the (proper brackets in) groupings has non-zero homology in dimension $n-3$ (its highest dimension).

Proof. Let $D = \mathrm{conv}(v_1, \ldots, v_{n+1})$ be a solid $(n+1)$-gon in $\mathbb{R}^2$. The association $[i,j] \leftrightarrow \overline{v_i v_{j+1}}$ gives a bijection between proper brackets and chords of D, such that groupings correspond to triangulations. For any $f : \{v_i\} \to \mathbb{R}$ with $f(v_1) = f(v_2) = f(v_3) = 0$, let $v_i' \in \mathbb{R}^3 = (v_i, f(v_i))$. Then $D_f = \mathrm{conv}(v_1', \ldots, v_{n+1}')$ is a polytope in $\mathbb{R}^3$ projecting down to D. The facets on the upper surface of D_f define a partial triangulation T_f of D and it is obvious that if $T_f = T_g$ then $T_{af+bg} = T_f = T_g$ where $a, b \geq 0$ and not $a=b=0$. Furthermore, every partial triangulation arises this way, since the conditions $f(v_1)=f(v_2)=f(v_3)=0$ can be imposed on an arbitrary f by adding a suitable linear function $g: \mathbb{R}^2 \to \mathbb{R}$. Since it is clear that the convex cone of functions corresponding to a given partial triangulation T is on the boundary of the cones of those yielding refinements of T, and also that the functions yielding any single chord form a 1-dimensional ray, this construction realizes a simplicial decomposition of S^{n-3} (the ray space of these functions) by the simplicial complex in question. $\blacksquare$

Remark. If D is any polytope, we may consider in place of triangulations, polytopal decompositions of D, that is, partitions of D into polytopes $E_1, \ldots, E_k$ meeting only along their boundaries, such that each E_i is the convex hull of a subset of D's vertices. There is a simplicial complex whose points are the refinement-minimal decompositions of D (not necessarily into two pieces!) and whose simplices are the maximal (= the simplicial) ones. The above construction shows this complex is also a topological sphere. Question: is it in general a polytope?

Now to construct the associahedron: for each grouping G let $v_G \in V$ be defined by the equations

$$\langle \ell_{[i,j]}, v_G \rangle = 1 \qquad [i,j] \in G\ ,\ i \neq j$$
$$\langle e_n, v_G \rangle = -1\ .$$

Since each v_G obeys $\langle \ell_{[1,n]}, v_G \rangle = 1 = \langle -e_n, v_G \rangle$, they all lie on an affine subspace W of dimension $n-2$. Let $\mathcal{P} = \bigcap\limits_{\substack{[i,j] \\ \text{proper}}} \{ w \in W \mid \langle \ell_{[i,j]}, w \rangle \leq 1 \}$.

Below we prove

<u>A3</u>. For $[i,j] \notin G$ proper, $\langle \ell_{[i,j]}, v_G \rangle < 1$.

It follows that

<u>Theorem</u> $\mathcal{P}$ is an associahedron with vertices v_G and facets supported by $\ell_{[i,j]} = 1$.

<u>Proof</u>. By A1 and A3, none of the defining conditions $\langle \ell_{[i,j]}, w \rangle \leq 1$ is redundant, so $\mathcal{P}$ does have a facet for each proper bracket, and these are all the facets by definition. A1 and A3 also show that each v_G is a vertex, and the incidence relation is correct

by definition. The only problem is to show there are no other vertices. But A2 shows there is a cocycle supported on $\{v_G\}$, and this is only possible if these are all the vertices. $\boxtimes$

The rest of these notes are devoted to proving A3. Define v_G^0 by

$$\langle \ell_{[i,j]}, v_G^0 \rangle = 1 \qquad [i,j] \in G,\ i \neq j$$
$$\langle e_n, v_G^0 \rangle = 0 .$$

<u>A4</u>. If $i \leq j < k$ and $[i,j], [i,k] \in G$ then:

$$\langle e_j, v_G^0 \rangle \underset{(1)}{=} (1+\delta)(1-\beta)\left[\left(1-\tfrac{\alpha}{1-\beta}\right)\langle e_k, v_G^0\rangle - \tfrac{\alpha}{1-\beta} + \delta\right]$$
$$\underset{(2)}{=} (1-\alpha)\left[\langle e_k, v_G^0\rangle + \delta\langle e_k, v_G^0 - v_G\rangle\right] + \alpha\langle e_k, v_G\rangle;$$
$$\langle e_j, v_G \rangle \underset{(3)}{=} (1+\delta)(1-\beta)\left[\left(1-\tfrac{\alpha}{1-\beta}\right)\langle e_k, v_G\rangle - \tfrac{\alpha}{1-\beta} + \delta\right]$$
$$\underset{(4)}{=} \beta\left[\langle e_k, v_G^0\rangle + \delta\langle e_k, v_G^0 - v_G\rangle\right] + (1-\beta)\langle e_k, v_G\rangle;$$
$$(\beta + \delta\beta)\langle e_j, v_G^0\rangle + (\alpha - \delta(1-\alpha))\langle e_j, v_G\rangle \underset{(5)}{=} -\alpha + \delta(1-\alpha);$$

where $\alpha = \begin{cases} \dfrac{\tau^{j-i+1}}{1+\tau^{j-i}} & \text{if } j+1<k \\ 0 & \text{if } j+1=k \end{cases}$, $\quad \beta = \dfrac{\tau^{j-i}}{1+\tau^{j-i}}$

$$\delta = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i<j \end{cases}$$

<u>Proof.</u> By reverse induction on j.

Equations (1) and (3) follow from the formula

$$\ell_{[i,k]} = \ell_{[i,j]} + (1+\tau^{j-i})e_j + \tau^{j-i+1}\ell_{[j+1,k]} - (1-\tau^{j-i+1})e_k$$

and the conditions

$$\langle \ell_{[i,k]}, v_G \rangle = 1$$

$$\langle \ell_{[i,j]}, v_G \rangle = \begin{cases} 1 & \text{if } i<j \\ -\langle e_j, v_G\rangle & \text{if } i=j \end{cases}$$

$$\langle \ell_{[j+1,k]}, v_G \rangle = \begin{cases} 1 & \text{if } j+1<k \\ -\langle e_k, v_G\rangle & \text{if } j+1=k \end{cases}$$

and the same conditions on v_G^0.

Equations (2) and (4) are both equivalent to (5) for e_k by rearranging and using $\delta^2 = \delta = 2\delta\beta$. If $k=n$, then (5) for e_k is trivial. If $k<n$, we have it by induction.

Finally, expanding $\langle e_j, v_G^0\rangle$ and $\langle e_j, v_G\rangle$ using (2) and (4) reduces (5) to (5) for e_k. ▨

A5. If $[i,k] \in G$, then $\langle e_k, v_G \rangle$ is determined once it is known which brackets containing $[i,k]$ lie in G.

Proof. A4 gives an explicit computation of it. ▨

A6. If $[i,k] \in G$, then once it is known which brackets contained in $[i,k]$ lie in G, each $\langle e_j, v_G \rangle$ $(i \le j \le k)$ is determined by $\langle e_k, v_G \rangle$, and each is a strictly increasing function of $\langle e_k, v_G \rangle$.

Proof. Also a corollary to A4. The increasing property is guaranteed because the coefficient $(1+\delta)(1-\beta)(1-\frac{\alpha}{1-\beta})$ is always positive (note that $\alpha+\beta<1$). ▨

A7. If $[i,k] \in G$, $i \le j < k$, then $\langle e_j, v_G \rangle > \langle e_k, v_G \rangle$.

Proof. Because $\alpha+\beta<1$, A4 (2) and (4) show that $\langle e_j, v_G^0 \rangle > \langle e_j, v_G \rangle$ for all $1 \le j \le n$. The result then follows by A4 (4). ▨

A8. Under the hypotheses of A4, $\langle e_j, v_G \rangle$ is strictly increasing with i and weakly decreasing with k.

Proof. Also a corollary to A4, once $\langle e_j, v_G^0 \rangle > \langle e_j, v_G \rangle$ for all j has been shown. ▨

A9. Let G, H be groupings such that for some $i < j \le k < m$, $[i,k] \in G$, $[j,m] \in H$, and G and H agree on all other brackets (in particular, $[j,k]$ and $[i,m]$ must be in both G and H).

Then for $i \le l \le j$:

$$\langle e_l, v_G \rangle > \langle e_l, v_H \rangle \qquad (6)$$

For $j \le l \le k$:

$$\langle e_l, v_G \rangle < \langle e_l, v_H \rangle \qquad (7)$$

For $k < l \le m$:

$$\langle e_l, v_G \rangle = \langle e_l, v_H \rangle. \qquad (8)$$

Proof. (8) is by ~~A5~~ A5 and A6. (7) is by A8 and A6. For (6) note that $\langle e_{j-1}, v_G \rangle > \langle e_{j-1}, v_H \rangle$ follows from A8 and $\langle e_k, v_G \rangle > \langle e_m, v_G \rangle = \langle e_m, v_H \rangle$ which follows from A7. Then (6) follows by A6. ▨

We are now ready to prove A3, which we restate as

<u>A10</u>. If $i<j$ and G maximizes $\langle \ell_{[i,j]}, v_G \rangle$, then $[i,j] \in G$.

<u>Proof</u>. Let $[i,k]$ be the largest bracket $[i,x]$ in G. If $i=1$, then $[i,k]=[1,n]$ and so $[i,j] \subseteq [i,k]$. If $i>1$, suppose $[i,j] \not\subseteq [i,k]$. Since $i>1$ and k is maximal, the bracket immediately containing $[i,k]$ is $[h,k]$ for some $h<i$. Since such brackets exist in G, let $[h',k]$ be the largest one. Since $k<j\leq n$, there is a bracket $[h',m] \in G$ with $m>k$. But now replacing $[h',k]$ by $[i,m]$ in G increases $\langle \ell_{[i,j]}, v_G \rangle$ by A9 and the fact that $\ell_{[i,j]}$ has positive coefficients on e_i through e_k. Since this is impossible, $[i,j] \subseteq [i,k]$.

Nearly the same argument shows G contains a bracket $[h,j]$ with $[i,j] \subseteq [h,j]$. Namely, if $j=n$ this is trivial, and if $j<n$ there

would othewise be brackets $[h,j]$, $[h,m]$, $[\ell,m]\in G$ with $i<h$, $\ell<h\leq j<m$. Replacing $[h,m]$ by $[\ell,j]$ in G increases $\langle e_j, v_G\rangle$ through $\langle e_{n-1}, v_G\rangle$ and decreases $\langle e_h, v_G\rangle$ through $\langle e_j, v_G\rangle$. But $\ell_{[i,j]}$ is negatively sensitive to $\langle e_h, v_G\rangle$ through $\langle e_j, v_G\rangle$. Specifically, if $h=j$ then $\ell_{[i,j]}$ restricts to $-e_j$ on these coordinates. Otherwise it restricts to $\tau^{h-i+1}\ell_{[h,j]}-(1-\tau^{h-i+1})e_j$, and the first term is constant. Therefore the replacement is impossible, and the maximal bracket $[h,j]$ contains $[i,j]$.

$[i,k]$ and $[h,j]$ are not disjoint, and they can be nested only if the smaller of them is equal to $[i,j]$. So $[i,j]\in G$. ▨